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**Homogenization problems  
in random media**

by

Dimitrios Kontogiannis

A dissertation submitted to the graduate faculty  
in partial fulfillment of the requirements for the degree of  
DOCTOR OF PHILOSOPHY

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Program of Study Committee:  
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## ABSTRACT

In this paper we study homogenization problems of partial differential equations in random domains. We give an overview of the classical techniques that are used to obtain homogenized equations over simple microstructures (for instance, periodic or almost periodic structures) and we show how we can obtain averaging equations over some particular random configurations. As it will be seen, such methods require ergodic theory, percolation, stochastic processes, in addition to the compactness of solutions and the convergence process. The structures considered can be used in microvascular modelling, sea ice pancake regions etc.

## CHAPTER 1. Introduction and Synopsis

Homogenization theory is related to the asymptotic behavior of partial differential equations describing physical phenomena in heterogeneous materials. In particular, we look for the effective (homogenized) equations that describe the characteristics of the inhomogeneous medium as the length of a small parameter  $\epsilon$  tends to zero. This small parameter is the length of the heterogeneity.

The classical approach to such problems is to consider two different scales, the macroscopic scale  $x$  on which the weighted equations should be described and the microscopic scale  $y$  which is related to the local structure of the heterogeneity. The two scales are related to each other with the ratio  $\epsilon = \frac{x}{y}$ . In this setting, we can say that  $x$  is the slow variable,  $y$  is the fast variable and  $\epsilon$  is associated with the local oscillations of the microstructure. Below we give a brief description of the periodic problem for elliptic equations and the common methods to obtain the effective equations. Several results on stochastic homogenization are also included to see the development of the theory up to the current state. We then propose stochastic modeling based on continuum percolation that allow us to obtain homogenized equations over random structures.

## CHAPTER 2. Periodic homogenization

We first discuss briefly the common methods of periodic homogenization. The stochastic version of some methods will also be introduced in the next chapter and will lead to the presentation of the current results. We focus on elliptic equations, although the results hold for parabolic equations as well.

### 2.1 The two scale asymptotic expansion

Let  $D \subset \mathbb{R}^n$  be an open set with boundary  $\partial D$ . We decompose  $D$  into a union of  $n$ -dimensional cubes and let  $Y = (0, l_1) \times (0, l_2) \times (0, l_3) \times \dots (0, l_n)$  be one representative cube. From  $Y$  we remove an open set  $F$ , with  $\bar{F} \subset \bar{Y}$ . Let  $G = Y \setminus F$ .

We denote by  $\epsilon F$  the  $\epsilon$ -homothety of the set  $F$  and by  $\tau(\epsilon F)$  the union of translations of  $\epsilon F$ .

Let us now denote by  $F^\epsilon = D \cap (\tau(\epsilon F))$ . and  $G^\epsilon = D \setminus F^\epsilon$ . Note that  $G^\epsilon$  is a set perforated periodically with holes. We will consider only holes and representative cells which fully belong to  $\Omega$ . Thus, the boundary  $\partial G^\epsilon$  consists of two parts: the interior boundary  $\partial_{\text{int}} G^\epsilon =$  (the boundary of the obstacles within  $D$ ) and the exterior boundary  $\partial_{\text{ext}} G^\epsilon = \partial G^\epsilon \setminus \partial_{\text{int}} G^\epsilon$ .

In  $G^\epsilon$  we consider the elliptic operator  $A^\epsilon = -\frac{\partial}{\partial x_i} \left( a_{ij}(x/\epsilon) \frac{\partial}{\partial x_j} \right)$ , where the coefficients  $a_{ij} \in L^\infty(\mathbb{R}^n)$  are  $Y$ -periodic and  $a_{ij} \xi_i \xi_j \geq \alpha |\xi|^2$ .

For  $f \in L^2(\Omega)$ , we consider the problem



$$\begin{aligned}
A^\epsilon u^\epsilon &= f, \quad x \in G^\epsilon \\
\frac{\partial u^\epsilon}{\partial \eta} &= 0, \quad x \in \partial_{\text{int}} G^\epsilon \\
u^\epsilon &= 0, \quad x \in \partial_{\text{ext}} G^\epsilon
\end{aligned} \tag{2.1.1}$$

We seek the asymptotic behaviour of the solution  $u^\epsilon$  as  $\epsilon \rightarrow 0$ . The classical approach for such problem is to assume that  $u^\epsilon$  can be written in the form

$$u^\epsilon = u_0(x, y) + \epsilon u_1(x, y) + \epsilon^2 u_2(x, y) + \dots, \tag{2.1.2}$$

where  $y = \frac{x}{\epsilon}$ ,  $u_j$  are periodic in  $y$ . Note that the derivative with respect to  $x$  becomes  $\nabla \rightarrow \nabla_x + \frac{1}{\epsilon} \nabla_y$ . Then, the elliptic operator takes the form

$A^\epsilon = \epsilon^{-2} A_2 + \epsilon^{-1} A_1 + \epsilon A_0$ , with

$$\begin{aligned}
A_0 &= -\frac{\partial}{\partial y_i} \left( a_{ij}(y) \frac{\partial}{\partial y_j} \right), \\
A_1 &= -\frac{\partial}{\partial y_i} \left( a_{ij}(y) \frac{\partial}{\partial x_j} \right) - \frac{\partial}{\partial x_i} \left( a_{ij}(y) \frac{\partial}{\partial y_j} \right), \\
A_2 &= -\frac{\partial}{\partial x_i} \left( a_{ij}(y) \frac{\partial}{\partial x_j} \right)
\end{aligned} \tag{2.1.3}$$

Clearly the operators  $A_i$ ,  $i = 1, 2, 3, \dots$  are all periodic in  $y$ . Substituting (2.1.3) into (2.1.1) and collecting the  $\epsilon^0$ ,  $\epsilon^1$ ,  $\epsilon^2$ ,  $\epsilon^3 \dots$  terms, we obtain a system of equations:

$$\begin{aligned}
A_0 u_0 &= 0 \\
A_0 u_1 &= -A_1 u_0 \\
A_0 u_2 &= -A_1 u_1 - A_2 u_0 + f
\end{aligned} \tag{2.1.4}$$

for the  $O(1)$ ,  $O(1/\epsilon)$ ,  $O(1/\epsilon^2)$  terms respectively. This first equation of system (2.1.4) implies that  $u_0$  is independent of  $y$ , that is  $u_0(x, y) = u(x)$ . Solving the system, with the help of Lax-Milgram theorem, we obtain the following result:

**Theorem 2.1.1.** *The solution  $u^\epsilon$  of (2.1.1) for small  $\epsilon > 0$  is approximately given by the solution of the boundary value problem*

$$\begin{aligned}\hat{A}u &= f, \quad x \in D \\ u &= 0, \quad x \in \partial D\end{aligned}\tag{2.1.5}$$

where  $\hat{A}$  consists of constant, averaged entries  $\hat{a}_{ij}$  given by the formula

$$\hat{a}_{ik} = \frac{1}{\text{mes}Y} \left[ \int_G a_{ik}(y) - a_{ij} \frac{\partial \chi^k}{\partial y_j} dy \right]\tag{2.1.6}$$

where  $\chi^k$  is the unique solution of the cell problem

$$\begin{aligned}A_0 \chi^k &= \frac{\partial a_{ij}}{\partial y_i}, \quad x \in G \\ \frac{\partial \chi^k}{\partial \nu} &= a_{ij}(y) \cos(n_y, y_i), \quad x \in \partial F\end{aligned}\tag{2.1.7}$$

We mention that the homogenized equations take into account only the interior boundary conditions and are not affected from the exterior boundary.

The two scale expansion has been widely used to treat various types of periodic structures. As a few references, we mention [4] and [8].

## 2.2 The two scale convergence

The two scale convergence was formulated and introduced by Allaire [2]. The advantage of this method is that it improves the justification of the first term  $u_0(x, y)$  of the expansion (2.1.2) and justifies at one step that the limit function  $u$  is the solution of the homogenized equation. The central definition for convergence of the method is the following:

**Definition 2.2.1.** *Let  $D$  be an open, bounded domain in  $\mathbb{R}^n$  and let  $Y = (0, 1)^n$  be the unit cube. Let  $C_{\text{per}}^\infty(Y)$  be the space of infinitely differentiable functions that are  $Y$ -periodic and  $\mathbb{D}(D; C_{\text{per}}^\infty(Y))$  the space of smooth compactly supported functions in  $D$  with values in  $C_{\text{per}}^\infty(Y)$ . We say that the family of solutions  $u^\epsilon \in L^2(D)$  two-scale converges to  $u_0(x, y) \in L^2(D \times Y)$  if*

$$\lim_{\epsilon \rightarrow 0} \int_D u^\epsilon(x) \phi(x, \frac{x}{\epsilon}) dx = \int_D \int_Y u_0(x, y) \phi(x, y) dx dy \text{ for all } \phi \in \mathbb{D}(D; C_{per}^\infty(Y)).$$

The compactness of solutions is guaranteed from the following theorem:

**Theorem 2.2.2.** *Every bounded sequence  $u^\epsilon$  in  $L^2(D)$  has a two-scale convergent subsequence that converges to a limit  $u_0(x, y) \in L^2(D \times Y)$ .*

Similar convergence results can be obtained in  $H^1(D)$ . The homogenization on elliptic equations follows from the following theorem [2]:

**Theorem 2.2.3.** *Let  $A(x, y)$  be an elliptic,  $Y$ -periodic diffusion tensor and let  $f(x) \in L^2(D)$ . The sequence of solutions  $u^\epsilon \in H_0^1(\Omega)$  of the problems*

$$\begin{aligned} -\nabla \cdot \left( A(x, \frac{x}{\epsilon}) \nabla u^\epsilon \right) &= f, \quad x \in D \\ u^\epsilon &= 0, \quad x \in \partial D \end{aligned} \tag{2.2.1}$$

*two-scale converges to the solution  $u \in H_0^1(D)$  of the homogenized problem*

$$\begin{aligned} -\nabla_y \cdot (A(x, y)(\nabla u(x) + \nabla_y u_1(x, y))) &= 0, \quad x \in D \times Y \\ -\nabla_x \cdot \left( \int_Y A(x, y)(\nabla u(x) + \nabla_y u_1(x, y)) dy \right) &= f(x), \quad x \in D \times Y \\ u &= 0, \quad x \in \partial D \end{aligned} \tag{2.2.2}$$

Stochastic versions for both methods have been introduced and will be provided in chapter 3.

## 2.3 The strange term of Cioranescu-Murat

An interesting result in the theory of homogenization was obtained in [7]. The authors considered the Dirichlet problem for the Laplace equation in periodically perforated domains. The main feature of their work is that the limit equation may vary, depending on the capacity of the holes. In terms of functional analysis, the following statements correspond to their results:

**Definition 2.3.1.** *A sequence of subsets  $D_n \subset D$  is called fading if for every sequence of bounded distributions  $\mu_n$ , with  $\|\mu_n\|_{H^{-1}(D_n)} \leq M$ ,  $\mu_n \rightarrow 0$  weakly.*

It is known that if there exists sequence  $\psi_n \in H^1(\mathbb{R}^n)$  such that  $\psi_n = 1$  on a neighborhood of  $D_n$  and  $\|\psi_n\|_{H^1(D_n)} \rightarrow 0$  as  $n \rightarrow \infty$ , then  $D_n$  is fading.

Their result can be summarized in the following theorem:

**Theorem 2.3.2.** *Let  $D_\epsilon \subset \mathbb{R}^n$  be a periodically perforated domain with holes  $T_i^\epsilon$  of radius  $\alpha_\epsilon$ ,  $i = 1, \dots, n(\epsilon)$  that remain away from the boundary of the periodic cell  $Y$ . Consider the problem*

$$\begin{aligned} -\Delta u^\epsilon &= f \text{ in } D_\epsilon \\ u^\epsilon &= 0 \text{ on } \partial T_\epsilon \end{aligned} \tag{2.3.1}$$

*Then, for any  $\epsilon > 0$ , the solution  $u^\epsilon$  of the problem (2.3.1) has an extension (by zero) in  $H_0^1(D)$ , still denoted by  $u^\epsilon$ , such that  $\{u^\epsilon; \epsilon \rightarrow 0\}$  remains bounded in  $H_0^1(D)$ . Depending on the ratio  $\frac{\alpha_\epsilon}{\epsilon^n}$  as  $\epsilon \rightarrow 0$ , three situations may occur:*

*a) If  $\alpha_\epsilon \ll \epsilon^n$ ,  $u^\epsilon \rightarrow u$  in  $H_0^1(D)$ , where  $u$  satisfies*

$$\begin{aligned} -\Delta u &= f \text{ in } D \\ u &= 0 \text{ on } \partial D \end{aligned} \tag{2.3.2}$$

*b) If  $\alpha_\epsilon \approx \epsilon^n$ ,  $u^\epsilon \rightharpoonup u$  in  $H_0^1(D)$  where  $u$  satisfies*

$$\begin{aligned} -\Delta u + \mu u &= f \text{ in } D \\ u &= 0 \text{ on } \partial D. \end{aligned} \tag{2.3.3}$$

*c) If  $\alpha_\epsilon \gg \epsilon^n$ ,  $u^\epsilon \rightarrow 0$  in  $H_0^1(D)$*

where  $\mu \in W^{-1,\infty}(D)$  is the so called strange term coming to the equation when the radius of the obstacles has the critical value  $\alpha^\epsilon = C\epsilon^{n/n-2}$  for  $n > 2$ , and is the strong limit in  $H^{-1}(D)$  of the sequence of measures  $\mu^\epsilon = \frac{(n-2)C^{n-2}}{1-C^{n-2}\epsilon^2} \sum_{i=1}^{n(\epsilon)} \epsilon \delta_i^\epsilon$  (Dirac measures supported on each sphere  $\partial T_i^\epsilon$ ). The method to obtain the limit equations is Tartar's energy method and the computation of this critical value for the radius is based on the fundamental solution of the Laplacian in an annulus surrounding the circular hole  $T^\epsilon$ . Convergence of the energy functional follows as well.

The same results hold for obstacles of different shape, but still periodically distributed. For more details, please see [7]. Other authors who have studied Dirichlet problems with small

holes are, among others, Papanicolaou and Varadhan [14], Sanchez-Palencia [15], Cioranescu [8], G.Allaire [3].

A summary of homogenization techniques can be found in [10], including the preceding methods as well as DeGiorgi's  $\Gamma$ –convergence,  $H$ -convergence and the energy method.

## CHAPTER 3. Stochastic homogenization

### 3.1 Ergodic theory

### 3.2 Ergodic theory

Stochastic homogenization is mainly based on the most generalized notion of periodicity which we call stationarity. Ergodic theory is related to the study of dynamical systems with an invariant measure. Applications of the theory can be found in statistical mechanics, probability theory, algebraic groups etc.

Let  $(\Omega, F, \mu, T)$  be a measure preserving dynamical system defined on a probability space with the following structure:  $F$  is the  $\sigma$ -algebra on  $\Omega$ ,  $\mu$  is the probability measure,  $T$  is the measure preserving transformation such that for any  $A \in F$ ,  $\mu(T^{-1}(A)) = \mu(A)$ . The system  $(\Omega, F, \mu, T)$  is ergodic if the  $\sigma$ -algebra of  $T$ -invariant events is trivial, that is, it occurs with probability zero or one. To see the importance of the theory on averaging problems, we state (among many versions) a subadditive ergodic theorem.

A function  $\mu : A \rightarrow \mathbb{R}$  is called subadditive if for every finite and disjoint family  $(A_i)_{i \in I}$  with  $|A \setminus \cup_{i \in I} A_i| = 0$ ,

$$\mu(A) \leq \sum_i \mu(A_i).$$

We say that  $\mu$  is dominated if  $0 \leq \mu(A) \leq C|A|$  for all sets  $A$ . Consider now the family of dominated, subadditive functions and the group of translations  $(\tau_z \mu)(A) = \mu(\tau_z A)$ , where  $\tau_z A = \{x \in \mathbb{R}^n : x - z \in A\}$ .

**Theorem 3.2.1.** (*Ergodic*)(see [1], [9]): Let  $\mu : \Omega \rightarrow \mathbb{R}^n$  be a subadditive process, periodic in law, in the sense that  $\mu(\cdot)$  and  $\tau_z \mu(\cdot)$  have the same distribution for every  $z \in \mathbb{Z}^n$ . Then, there exists measurable function  $\phi : \Omega \rightarrow \mathbb{R}$  and a subset  $\Omega' \subset \Omega$  of full measure such that

$\lim_{t \rightarrow \infty} \frac{\mu(\omega)(tQ)}{|tQ|} = \phi(\omega)$  exists a.e.  $\omega \in \Omega'$  and for every cube  $Q \subset \mathbb{R}^n$ . Furthermore, if  $\mu$  is ergodic then  $\phi$  is constant.

### 3.3 Some results on stationary environments

The importance of the ergodic theory is that we can still average even if periodic behaviour for the medium is replaced with equiprobable behaviour. Papanicolaou and Varadhan[14] showed that if  $a(x)$  is a stationary random function, the sequence of solutions of the Dirichlet problems

$$\begin{aligned} -\nabla \cdot \left( a \left( \frac{x}{\epsilon}, \omega \right) \nabla u^\epsilon \right) &= f \text{ in } D \\ u^\epsilon &= 0 \text{ on } \partial D \end{aligned} \quad (3.3.1)$$

converge in the mean square to the solution of the deterministic equation

$$\begin{aligned} -\sum_{i,j=1}^n q_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} &= f \text{ in } D \\ u &= 0 \text{ on } \partial D \end{aligned} \quad (3.3.2)$$

in the sense that the  $L^2$ - norm of the expectation of  $u^\epsilon - u$  converges to zero.

The stochastic version of the two-scale convergence is introduced in [5]. Stochastic vector calculus and the corresponding  $L^p$ -theory is used for this purpose. The convergence process is similar in this case, as long as the two scale product  $\int dx dy$  is replaced with  $\int dx d\mu$ , where  $\mu$  is the probability measure:

**Definition 3.3.1.** Let  $Q \subset \mathbb{R}^n$  be bounded and let  $\{u^\epsilon\}$  be a sequence in  $L^p(Q \times \Omega)$  where  $\Omega$  is a probability space. We say that the sequence  $\{u^\epsilon\}$  stochastically two-scale converges to  $u \in L^p(Q \times \Omega)$  if for each  $\psi \in L^q(Q \times \Omega)$ ,

$$\lim_{\epsilon \rightarrow 0} \int_{Q \times \Omega} u^\epsilon(x, \omega) \psi(x, T(\epsilon^{-1}x)\omega) dx d\mu = \int_{Q \times \Omega} u(x, \omega) \psi(x, \omega) dx d\mu \quad (3.3.3)$$

Both the deterministic and stochastic versions [5] of two-scale convergence have been used for elliptic-parabolic equations, as well as the derivation of Darcy's law from Stokes equations.

We refer a paper from Dal Maso-Modica [9] for the use of ergodic theory in the calculus of variations, according to the following setting:

Define the translation operator  $\tau_z$  that acts through the following relations:

$$\tau_z u(x) = u(x - z), \text{ and } \tau_z A = \{x \in \mathbb{R}^n : x - z \in A\}, \text{ and the homothety operator } (\rho_\epsilon F)(u, A) = \epsilon^n F(\rho_\epsilon u, \rho_\epsilon A) \text{ where } (\rho_\epsilon u)(x) = \frac{1}{\epsilon} u(\epsilon x), (\rho_\epsilon A) = \{x \in \mathbb{R}^n : \epsilon x \in A\}.$$

A stochastic homogenization process is a family of random variables  $(F_\epsilon)_{\epsilon>0}$  on a probability space  $(\Omega, F, P)$  that has the same distribution law with the random functionals given by  $[(\rho_\epsilon F)(\omega)(u, A)]$  for  $u \in W^{1,p}(A)$ . This means

$$P\{\omega \in \Omega : F_\epsilon(\omega) \in S\} = P\{\omega \in \Omega : \rho_\epsilon F(\omega) \in S\} \text{ for any open set } S.$$

Furthermore, we say that the random functional  $F$  is stochastically periodic, that is,  $F$  has the same distribution law as the random functional  $(\tau_z F)(\omega)(u, A) = F(\tau_z u, \tau_z A)$ .

**Theorem 3.3.2.** *Let  $F(\omega)(u, A) = \int_A f(x, \nabla u) dx$ , where  $f$  satisfies standard growth conditions:  $k|p|^2 \leq f(x, p) \leq K|p|^2$  for some positive constants  $k, K$ . Denote the minimizer of  $F$  by  $m(F, u_0, A) = \min_u \{F(u, A) : u - u_0 \in W_0^{1,2}(A)\}$ .*

*If  $F$  is a random integral functional and if  $F$  and  $\tau_z F = F(\tau_z u, \tau_z A)$  have the same distribution law, then the limit  $\lim_{t \rightarrow \infty} \frac{m(F(\omega), u_0, Q_t)}{|Q_t|}$  exists. If in addition  $F$  is ergodic, the limit is constant.*

The proof of this theorem is based on the ergodic theorem (see [1] or theorem (3.1.1)), since the function  $\mu(\omega)(A) = m(F(\omega), l_p, A)$  is dominated and subadditive.

We also mention a recent paper by Caffarelli-Mellet [6] in which the authors extended the results of Cioranescu-Murat in the case that the obstacle problem is considered in a domain perforated by balls of random radius centered at the  $\mathbb{Z}^n$  lattice. As they showed, depending on the capacity of the holes, we still have an additional term that appears in the averaged equations. We will see in chapter 5 that their result is easily generalized if we simply replace the periodic lattice with a set of random points, as long as the spheres are non-intersecting.

The same averaged equations can be derived for more general heterogeneous regions. Please see Appendix for some qualitative characteristics of such regions. We show how the ergodic theory can be used for to provide the limit equations as long as the random structures are modeled in such way that some quantitative characteristics of the domains can be measured in the limit  $\epsilon \rightarrow 0$ . For this purpose, we use models of continuum percolation and we show how



the theory of percolation can be applied in homogenization. For this theory, please see [13].

## CHAPTER 4. Percolation theory and Random modelling

Percolation theory deals with the behavior of connected elements in random graphs. A cluster is a simply connected group of elements. Percolation phenomena arise in transport, porous media, spread of diseases, conductivity problems, sea ice etc. The theory was introduced by Broadbent and Hammersley in 1957, when they considered the problem of fluid flow through a porous medium formed by channels, keeping in mind that some of the channels may be blocked.

In the discrete version of percolation, we consider the  $\mathbb{Z}^n$  lattice and for  $p \in [0, 1]$  we connect the point  $x \in \mathbb{Z}^n$  to one of its  $2n$  nearest neighbors with probability  $p$ , independently of the other points. We describe distances in the lattice in the following sense: two vertices  $x, y \in \mathbb{Z}^n$  are neighbors if  $|x - y| = 1$ . We can also define boundary  $\partial Q_n$  of the sets  $Q_n = [1, n]^d \cap \mathbb{Z}^n$ , where  $\partial Q_n = \{y \in \mathbb{Z}^d : \exists x \in Q_n : |x - y| = 1\}$ .

According to this setting, each pair of neighbours has an edge between them with probability  $p$ . The edge is also called a bond. A path is a finite or infinite alternating sequence  $(z_1, e_1, z_2, e_2, \dots)$  of vertices  $z_i$  and bonds  $e_i$  such that  $z_i \neq z_j$  and  $e_i \neq e_j$  for  $i \neq j$ . Two vertices are connected if there is a finite open path from one to the other. An open cluster is a set of connected vertices that is maximal with respect to this property. It can be either finite or infinite.

The main question that arises is if there exists a critical threshold of  $p$  at which an infinite cluster occurs. In many cases, like the two-dimensional lattices, the critical value  $p_c$  can be computed explicitly. This critical probability is an increasing function of  $p$ .

## 4.1 Continuum Percolation

A more general situation appears in the models of continuum percolation, where the integer lattice is replaced by a random set of points in  $\mathbb{R}^n$ . The random positions are usually formed by the realization of a point process. Point processes are important models and have been used in a variety of problems such as environmental modeling, air pollution, weather radar images, traffic networks, statistical mechanics etc. For a complete account of the theory, please see [13].

A point process is thought as a random set of points in the space. In particular we give the following definition:

**Definition 4.1.1.** *Let  $B$  be a Borel subset of  $\mathbb{R}^n$  and denote by  $N$  the set of all counting measures in  $\mathbb{R}^n$ . Let  $\psi \in N$  be a counting measure, i.e. a measure which is 1 on each point  $x \in B$ . Then  $N$  is identified as the set of all such configurations of points in  $\mathbb{R}^n$  without limit points. According to this setting,  $\psi(A)$  = random number of points in  $A$ , for any set  $A \subset \mathbb{R}^n$ . Then, a point process is defined as a measurable map  $X$  from a probability space  $(\Omega_1, F, P)$  into  $(N, M, P)$ , where  $M$  is the corresponding  $\sigma$ -algebra.*

The periodicity of the lattice-type structures is replaced with the assumption that the point process  $X(\cdot)$  is stationary:

If  $T_\alpha$  is the translation in  $\mathbb{R}^n$  by a vector  $\alpha$ ,  $T_\alpha(\beta) = \beta + \alpha$ ,  $\forall \beta \in \mathbb{R}^n$ , then  $T_\alpha$  induces a transformation  $S_\alpha : N \rightarrow N$  through the operation  $(S_\alpha \psi)(A) = \psi(T_\alpha^{-1}A)$   $\forall A \in B^n$  and similar operation holds for the set-measures.

**Definition 4.1.2.** *The point process  $X$  is stationary if its distribution is  $S_\alpha$ -invariant for any  $\alpha \in \mathbb{R}^n$ .*

Two common models in the theory of continuum percolation are the Boolean model and the random connection model. Both models are based on occurrences of Poisson processes. From this point, we assume that the point process  $X$  is a Poisson process of density  $\lambda$ :

(i) for any collection of mutually disjoint sets  $A_1, A_2, \dots, A_k$ ,

the random variables  $X(A_1), X(A_2), \dots, X(A_k)$  are mutually independent, and

(ii) for any bounded set  $A \subset B^n$  and any non-negative integer  $k \geq 0$ ,  $P(X(A) = k) = e^{-\lambda(A)} \frac{\lambda^k (l(A))^k}{k!}$ , where  $l(\cdot)$  is the Lebesgue measure.

#### 4.1.1 The Boolean model

The Boolean model is driven by a Poisson process  $X$  and each point of the process is the center of a ball of random radius. The region in the space that is covered by at least one ball is called the occupied region and its complement is the vacant region. To be able to have shifting properties, we construct the models as follows. The Poisson process is defined in a probability space  $(\Omega_1, F_1, P_1)$  and we consider a second space  $\Omega_2 = \prod_{\psi \in N} \prod_{z \in Z^n} [0, \infty)$  equipped with the probability measure  $\mu$  on  $[0, \infty)$ . Setting  $\Omega = \Omega_1 \times \Omega_2$  with product measure  $P = P_1 \times P_2$ , the Boolean model is defined as the map  $(\omega_1, \omega_2) \rightarrow (X(\omega_1), \omega_2)$  from  $\Omega$  into  $N \times \Omega_2$ .

According to this construction, the radii of the balls are independent of the point process and we obtain the shifting properties that ergodic theory requires.

We use the notation of [13] and we denote by  $(X, \rho, \lambda)$  the Boolean model obtained from a Poisson process  $X$  of density  $\lambda$  and radius random variable  $\rho$ .

#### 4.1.2 The Random connection model

As in Boolean models, the Poisson process is the first characteristic of the model and it assigns randomly points in the space. The second characteristic of the model is the connection function, which plays an essential role to the model and the homogenization process as we will see later. A connection function  $g : \mathbb{R}^+ \rightarrow [0, 1]$  connects two points  $x_1, x_2 \in X$  with probability  $g(x_1 - x_2)$ , where  $\cdot$  denotes the Euclidean distance. Depending on the construction that we need, we can choose  $g$  with specific characteristics. For example, we may assume that  $g$  is decreasing with respect to the distance. Such models are defined in product spaces as before. We denote them by  $(X, g, \lambda)$ .

**Definition 4.1.3.** *Two points  $x, y$  of the process are connected if there is a sequence of points  $\{x_0 = x, x_1, x_2, \dots, x_n = y\}$  such that the line segment  $\{x_i, x_{i+1}\}$  is open for all  $i = 1, \dots, n-1$ .*

As in the discrete percolation, a component is a set of points such that any two points of this set are connected and the set is maximal with respect to this property.

### 4.1.3 Ergodic properties of point processes

The one dimensional ergodic theorem is stated as follows (see for instance [13]):

**Theorem 4.1.4.** *Let  $(\Omega, F, \mu, T)$  be a measure preserving dynamical system and let  $f$  be  $\mu$ -integrable function on  $\Omega$ . Then,  $\frac{1}{n} \sum_{i=0}^{n-1} f(T^i(\omega)) \rightarrow E(f|I)(\omega)$ , as  $n \rightarrow \infty$  a.s. where  $I$  is the  $\sigma$ -algebra of  $T$ -invariant sets.*

For point processes, we identify any element  $\omega \in \Omega$  with a counting measure. Then the shift by distance  $t$ ,  $T_t$ , induces a transformation through  $(S_t\omega)(A) = \omega(T_t^{-1}A)$  for all measurable sets  $A \in \mathbb{R}$ .

**Definition 4.1.5.** *A stationary point process is ergodic if the group  $\{S_x : x \in \mathbb{R}^n\}$  acts ergodically on  $(\Omega, F, \mu)$ .*

An important aspect of continuum percolation models is that the ergodic properties of the process  $X$  are carried over the two models. The following results are well known [13]:

**Theorem 4.1.6.** *: A Poisson point process is stationary ergodic.*

**Theorem 4.1.7.** *Suppose that the point process  $X$  is ergodic. Then the random connection model  $(X, g, \lambda)$  and the Boolean model  $(X, \rho, \lambda)$  are also ergodic.*

Let  $(\Omega_1, F_1, P_1)$  be a probability space associated with the Poisson process  $X$  and let  $\omega \in \Omega_1$  be a point configuration on  $\mathbb{R}^n$  that is assumed to be locally finite and countable. This means that we have a finite number of points hits every compact set  $K \subset \mathbb{R}^n$  almost surely:  
 $P(\omega \in \Omega : \psi(K) < \infty \text{ for all compact } K \subset \mathbb{R}^n) = 1.$

## 4.2 Construction of random domains for homogenization

### 4.2.1 Random connection models

We start using the random connection model  $(X, g, \lambda)$  in the following way:

Suppose that  $\omega$  is a given realization for  $X$  which is locally finite and let  $x_i \in X$  be a given point of this realization.

Consider the annulus  $A = \{x \in \mathbb{R}^n : c_1 \leq x - x_i \leq c_2\}$ , where  $c_1, c_2$  are positive constants with  $c_1 \leq c_2$ .

We want to connect the point  $x_i$  with all the points in  $A$  that are given from  $X$ . For this purpose we choose the connection function

$$g(|x - x_i|) = \begin{cases} 1 & \text{if } c_1 \leq x - x_i \leq c_2 \\ 0 & \text{otherwise} \end{cases}$$

For a point  $x_j \in A$ , we denote by  $l_{ij}(\omega) = l(x_i, x_j)$  the line segment with endpoints  $x_i, x_j$  and let  $T_{c_1/2}(l_{ij})(\omega)$  the tube of radius  $c_1/2$  surrounding  $l_{ij}$ . Let now  $T(x_i)(\omega) = \cup_j T_{c_1/2}(l_{ij})(\omega)$  and  $F(\omega, c_1/2) = \cup_i T(x_i)(\omega)$  for all points  $x_i$  of the process.

Thus, the set  $F(\omega, c_1/2)$  is the union of random tubes obtained from the given realization of the point process. Let  $G(\omega, c_1/2) = \mathbb{R}^n \setminus F(\omega, c_1/2)$ .

We define the indicator function

$a(\omega, x) = 1 - \min\{X_{F(\omega, c_1/2)}, 1\}$  which is zero in the the union of tubes and one elsewhere.

Let  $D$  be an open, bounded domain of  $\mathbb{R}^n$  and consider the random functional  $F(\omega)(u, D) = \int_D a(\omega, x) |\nabla u|^2 dx = \int_{G(\omega) \cap D} |\nabla u|^2 dx$  for  $u \in W^{1,2}(D)$ . This functional is periodic in law and independent at large distances, thus ergodic.

Furthermore let  $(\rho_\epsilon F)(u, A) = \epsilon^n F(\rho_\epsilon u, \rho_\epsilon A)$  where  $(\rho_\epsilon u)(x) = \frac{1}{\epsilon} u(\epsilon x)$ ,  $(\rho_\epsilon A) = \{x \in \mathbb{R}^n : \epsilon x \in A\}$ . Then the family

$$F^\epsilon(u, D) = \rho_\epsilon F(u, D) = \int_{\epsilon G(\omega) \cap D} |\nabla u|^2 dx$$

satisfies the assumptions of theorem 3.2.2. Note that the  $\rho_\epsilon$ -homothetic functional is the functional obtained if we scale by  $\epsilon$  the distance between the connected points of the set  $F(\omega, c_1/2)$  that corresponds to the union of tubes  $\epsilon F = F(\epsilon\omega, \epsilon c_1/2)$ , where  $\epsilon\omega$  maps to the point measure whose support is  $\{\epsilon x_i\}$  and  $\{x_i\}$  is the support of  $X(\omega)$ . Note that the scaling properties of this model are the same (in terms of distribution) with the model that we have if we choose

$$g_\epsilon(|x - x_i|) = \begin{cases} 1 & \text{if } c_1\epsilon \leq |x - x_i| \leq c_2\epsilon \\ 0 & \text{otherwise} \end{cases}$$

with density function  $\lambda/\epsilon$ .

Let us define  $F^\epsilon(\omega) = \epsilon F = F(\epsilon\omega, \epsilon c_1/2)$  and  $G^\epsilon(\omega) = \epsilon G(\omega) = \mathbb{R}^n \setminus F^\epsilon(\omega)$ .

According to this model, points that are too close to each other (with respect to the  $\epsilon$  scale) cannot be connected and the same holds for points that are too far. On the other hand, the points within a particular range are connected.

Suppose that we apply the described model in  $D$ . The picture that we obtain is a random set of line segments in  $D$ , with restricted length. Figure 4.1 shows a general random model.

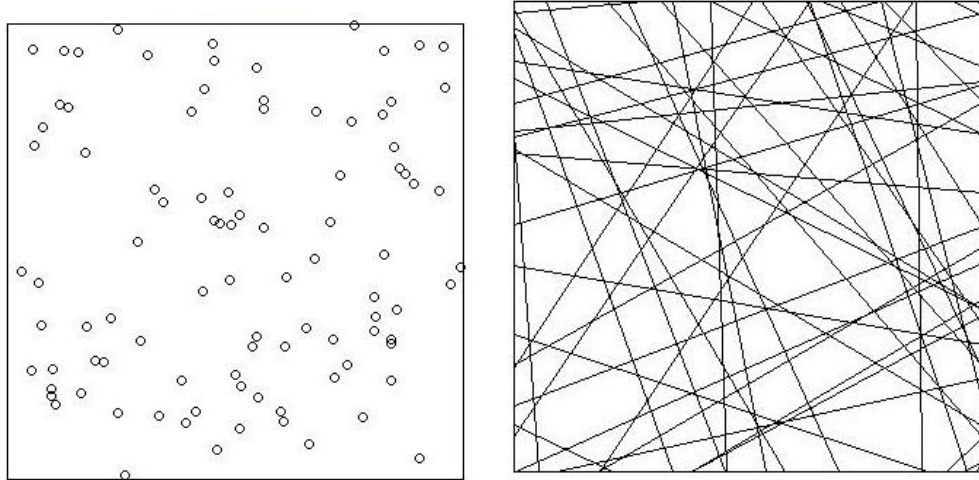


Figure 4.1 Realization of a point process and of a random connection model in the statistical package R

**Remark 4.2.1.** *The radius of the tubes need not necessarily be constant. We may consider the tubes  $T_{\rho_\epsilon(\omega)}(l(x_i, x_j)) = \{x \in \Omega : d(x, l(x_i, x_j)) \leq \rho_\epsilon(x, \omega)\}$ , where  $\rho_\epsilon(x, \omega)$  is continuous function on  $x$ , stationary ergodic and for some positive constants  $\rho_{1,\epsilon}, \rho_{2,\epsilon}$ ,  $\rho_{1,\epsilon} \leq \rho_\epsilon(x, \omega) \leq \rho_{2,\epsilon}$ . This is due to the fact that the product of two ergodic processes is also ergodic.*

### 4.2.2 Boolean models

We take again a probability space  $(\Omega_1, F_1, P_1)$  associated with the Poisson process  $X$  and let  $\omega \in \Omega$  be a given point configuration on  $\mathbb{R}^n$ . We consider a second space  $\Omega_2 = \prod_{\psi \in N} \prod_{z \in Z^n} [0, \infty)$  equipped with the probability measure  $\mu$  on  $[0, \infty)$  for the sequence of independent, identically distributed random radii. Let us denote by  $\bar{\omega} = (\omega, \bar{r})$  the realization of this Boolean model in  $\Omega = \Omega_1 \times \Omega_2$ , where  $\bar{r} = (r_1, r_2, \dots)$ .

Let  $F(\omega)$  be the union of random spheres obtained from the given realization of the point process. Let  $G(\omega) = \mathbb{R}^n \setminus F(\omega)$ . We define the indicator function

$$a(\omega, x) = 1 - \min\{X_{F(\omega)}, 1\} \text{ which is zero in the union of spheres and one elsewhere.}$$

Consider the functional

Let  $D$  be an open, bounded domain of  $\mathbb{R}^n$  and consider the random functional  $F(\omega)(u, D) = \int_D a(\omega, x) |\nabla u|^2 dx = \int_{G(\omega) \cap D} |\nabla u|^2 dx$ . This functional is periodic in law and independent at large distances, thus ergodic.

Again, the family  $\rho_\epsilon F(\omega)(u, D) = F^\epsilon(u, D)$ , defined as in section 3.2 satisfies the assumptions of theorem 3.2.2.

Depending on the problem under consideration, we may or may not allow the balls to intersect. For simplicity, we will consider only domains at which the balls are non intersecting and have a minimal positive distance between them. One way to model such case is to combine the Boolean and the random connection model in the following way:

Suppose the RCM  $(X, g, \lambda)$  is applied on a bounded region of  $\mathbb{R}^n$ . We want to assign every endpoint of the line process as the center of a ball of random radius. For fixed  $\epsilon > 0$  there is a set of points from the point process  $X$ . In our case, instead of constructing tubes, we let every point be the center of a ball with radius  $\rho(\omega) \leq \min d(x_i, x_j)(\omega)$ , where the minimum is taken over all the pairs of points  $x$  of  $X(\omega)$ . Note that, without any affect to our proofs, we may assume that  $\rho(\omega)$  is indentially distributed random variable taking maximum value  $\min d(x_i, x_j)(\omega)/4$ . We consider for simplicity the first case. According to this construction, we obtain a domain randomly perforated with balls of radius and with minimal distance between them.



Let us define  $F^\epsilon(\omega) = \bigcup_{i \geq 1} B(\epsilon \rho(\omega), \epsilon x_i) \cap D$  and  $G^\epsilon(\omega) \cap D = D \setminus F^\epsilon(\omega)$ . Note that  $\text{mes} F^\epsilon(\omega)$  tends to zero as  $\epsilon \rightarrow 0$ .

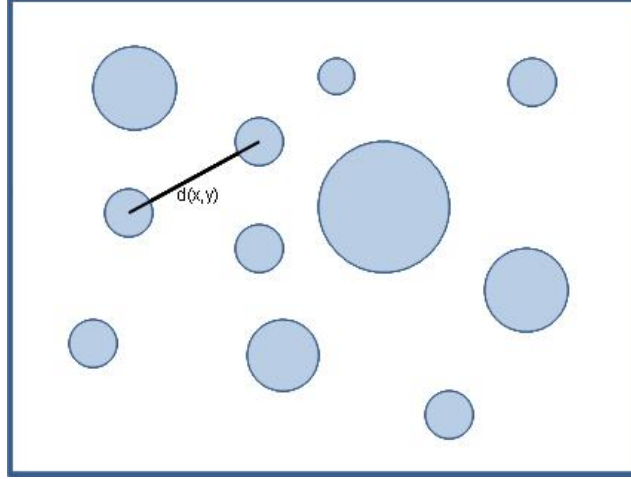


Figure 4.2 Boolean model with non-intersecting balls

## CHAPTER 5. Homogenization in random structures

### 5.1 The Dirichlet problem

#### 5.1.1 Boolean models and domains with fine-grained boundary

Let  $\omega \in \Omega_1$  be such that  $X(\omega)$  is locally finite.

We consider the Dirichlet problem of the form

$$\begin{aligned} \Delta u^\epsilon - \lambda u^\epsilon &= f, \quad x \in G^\epsilon(\omega) \cap D \\ u^\epsilon &= 0, \quad x \in \partial G^\epsilon(\omega) \end{aligned} \tag{5.1.1}$$

for  $u^\epsilon \in W^{1,2}(G^\epsilon(\omega))$ ,  $f \in L^2(D)$ . Here,  $G^\epsilon(\omega)$  is the domain perforated by non intersecting balls that we modelled at the end of chapter 4.

To define capacity characteristics for the massiveness of  $B_i^\epsilon(\omega)$ , let us consider the quantity

$$\text{cap}(B) = \inf_v \int_{\mathbb{R}^n} |\nabla v|^2 dx$$

for closed, bounded sets  $B$  in  $\mathbb{R}^n$ , over all functions  $v \in C_0^\infty(\mathbb{R}^n)$  taking value 1 in  $B$ . We consider only the case  $n \geq 3$  so that  $\text{cap}(B)$  defines the Newton capacity. Note that the capacity is invariant with respect to translations and rotations. In addition, if  $B_\epsilon$  is the  $\epsilon$ -homothetic contraction of  $B$ ,  $\text{cap}(B_\epsilon) = \epsilon^{n-2} \text{cap}(B)$ .

We also note that, clearly, as  $\epsilon \rightarrow 0$ , the diameter of the balls tend to zero. Now, note that the limits  $\lim_{\epsilon \rightarrow 0} \sum_{(D)} \text{cap}(B_i^\epsilon(\omega)) = \text{cap}(G^\epsilon(\omega)) = C$  exists due to the ergodic properties of the model and theorem 3.2.2 (see also section 4.2.2). The sum is taken over all balls strongly contained in  $D$ .

We now state the homogenization theorem for the Dirichlet problem related to Boolean models.

**Theorem 5.1.1.** *The family of solutions  $u^\epsilon$  of the Dirichlet problem 5.1.1 (extended by zero in  $\bigcup_{i=1}^{n(\epsilon)} B_i^\epsilon(\omega)$ ) converges in  $L^2(D)$  to the solution  $u$  of the boundary value problem*

$$\begin{aligned} \Delta u - (\lambda + C)u &= f, \quad x \in D \\ u &= 0, \quad x \in \partial D \end{aligned} \tag{5.1.2}$$

The proof of this theorem can be found in [12] under general assumptions which have been proved in our cases. See also section 5.1.3 for a relevant, but slightly different, proof for biharmonic operators.

### 5.1.2 RCM and connected domains

Let  $F^\epsilon(\omega)$  be the random set of channels that we constructed in section 4.2. We assume that the Poisson process  $X$  is locally finite in the sense that a finite number of points hits every compact set  $K \subset \mathbb{R}^n$  almost surely. That is,  $P(\omega \in \Omega : \psi(K) < \infty \text{ for all compact } K \subset \mathbb{R}^n) = 1$ .

Let  $G^\epsilon(\omega) = D \setminus F^\epsilon(\omega)$ . We want to show that for sufficiently small  $\epsilon$ , the volume of  $G^\epsilon(\omega)$  is strictly positive with probability 1. That is, there is  $\hat{\epsilon}(\omega) > 0$  such that

$$P(\omega \in \Omega : |G^\epsilon(\omega)| > 0 \text{ for all } \epsilon < \hat{\epsilon}(\omega)) = 1$$

For simplicity and without loss of generality, suppose that  $n = 2$  and  $D$  is a square of size  $1/\epsilon > 0$ . Consider a partition of  $D$  into squares  $D_i$ ,  $i = 1, \dots, 1/\epsilon^2$  of size 1. Suppose that the point process  $X$  is applied in  $D$ . Then, the probability of having zero points in a given  $D_i$  is  $e^{-\lambda} > 0$ . According to the law of large numbers, if  $1_i(\omega)$  is the indicator function of the empty square  $D_i$  of the partition,

$$P(\omega \in \Omega : \lim_{\epsilon \rightarrow \infty} \frac{1}{1/\epsilon^2} \sum_{i=1}^{\epsilon^2} 1_i(\omega) = e^{-\lambda}) = 1$$

which means that, in the limit, there are empty squares with probability one, as needed.

Since the points, within an  $\epsilon$ -range, are connected, the set

$K^\epsilon(\omega) = \bigcup_{\{i,j\}} T_{\rho_\epsilon}(l_{ij}) \cap \{\bigcup_{\{k,l\} \neq \{i,j\}} T_{\rho_\epsilon}(l_{kl})\}$  (the intersection of the tubes) is non-empty,  $K^\epsilon(\omega) \subset F^\epsilon(\omega)$ . Clearly, as  $\epsilon \rightarrow 0$ ,  $\text{mes} F^\epsilon(\omega) \rightarrow 0$  and  $G^\epsilon(\omega)$  becomes denser in  $\Omega$ . We can then consider two different homogenization problems: the first is when the elliptic equation is defined in  $G^\epsilon(\omega)$  for  $u^\epsilon \in W^{1,2}(G^\epsilon(\omega))$  and the second is the problem of decreasing volume when  $u^\epsilon \in W^{1,2}(F^\epsilon(\omega))$ .

We consider the Dirichlet problem of the form

$$\begin{aligned} -\Delta u^\epsilon + \lambda u^\epsilon &= f, \quad x \in G^\epsilon(\omega) \\ u^\epsilon &= 0, \quad x \in \partial G^\epsilon(\omega) \end{aligned} \tag{5.1.3}$$

for  $u^\epsilon \in W^{1,2}(G^\epsilon(\omega))$ ,  $f \in L^2(D)$  and as  $\epsilon \rightarrow 0$ ,  $G^\epsilon(\omega)$  is approximately  $D$ . We assume that for  $n \geq 3$  both  $G^\epsilon(\omega)$  and  $F^\epsilon(\omega)$  are connected.

Standard elliptic theory gives the existence of solutions. We extend the functions  $u^\epsilon$  by zero in  $F^\epsilon(\omega)$  and we keep the same notation for the extended sequence of functions. We denote by  $Q_h^x$  the  $n$ -cube centered at  $x$  of length  $h$ ,  $\text{diam} D \gg h \gg \epsilon$ , and we define the local capacity functional

$$\text{cap}(x, h, \epsilon, \gamma, \omega) = \inf_{u^\epsilon} \int_{Q_h^x} |\nabla u^\epsilon(y)|^2 + h^{-2-\gamma} |u^\epsilon(y) - 1|^2 dy \tag{5.1.4}$$

over all  $u^\epsilon \in W^{1,2}(Q_h^x) : u^\epsilon = 0$  in  $F^\epsilon(\omega)$ . Clearly, if  $F^{\epsilon_1}(\omega) \cap Q_h^x \subset F^{\epsilon_2}(\omega) \cap Q_h^x$ , then  $\text{cap}(x, h, \epsilon_1, \gamma, \omega) \leq \text{cap}(x, h, \epsilon_2, \gamma, \omega)$ . Thus, the capacity functional measures the massiveness of  $F^\epsilon(\omega)$  in  $\Omega$ . Note that, from our construction and due to theorem (4.1.5),  $F^\epsilon(\omega)$  and its complement are periodic in law and ergodic in the sense that disjoint cubes have the same distribution.

The main consideration for the capacity functionals (as well as the mean conductivity tensor in Neumann problems) that can be seen within the convergence part is that the flux in  $Q_h^x$  is given from the gradient of the function  $v^\epsilon$  which minimizes the integral

$$F_h^\epsilon(\omega) = \int_{G^\epsilon(\omega) \cap Q_h^z} |\nabla v^\epsilon|^2 dx$$

under the condition

$$\frac{1}{h^{2+\gamma}} \int_{G^\epsilon(\omega) \cap Q_h^z} |v^\epsilon - 1|^2 dx = o(h^n) \text{ as } h \rightarrow 0$$

as  $\epsilon > 0$  remains sufficiently small compared to  $h > 0$ .

Under this consideration,

$$\text{cap}(x, h, \epsilon, \gamma, \omega) = v^\epsilon + o(h^n)$$

Note that  $F_h^\epsilon(\omega) = \int_{Q_h^x} a(x, \omega) |\nabla v^\epsilon|^2 dx$ , where  $a(x, \omega) = 1 - \min\{1, X_{T_\rho(l_{ij})}\}$  which is 0 in the union of random channels with endpoints the points of the Poisson process and 1 elsewhere, as in section 4.2.1.

We will assume that theorem 3.2.2 is applicable over the class of functions in  $W^{1,2}(D)$  such that the last condition is satisfied. Hence, we will assume that the limit  $c = \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{v^\epsilon}{h^n}$  exists almost all  $\omega \in \Omega_1$ .

Due to the last considerations, we assume that the limit

$$c = \lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{\text{cap}(x, h, \epsilon, \gamma, \omega)}{h^n}$$

exists.

Furthemore, we assume that

$$\limsup_{\epsilon \rightarrow 0} \frac{\text{cap}(x, h, \epsilon, \gamma, \omega)}{h^n} < A$$

for all  $x \in D$  with  $A$  independent of  $h$ .

Our main homogenization theorem is the following:

**Theorem 5.1.2.** *Let  $F^\epsilon(\omega)$  be the sequence of RCM domains constructed in section 4.2.1 and  $G^\epsilon(\omega)$  be its complement set. Let  $u^\epsilon \in W^{1,2}(G^\epsilon(\omega))$  be the family of solutions of the boundary value problems (5.1.1) extended by zero in  $F^\epsilon(\omega)$ . Then, as  $\epsilon \rightarrow 0$ ,  $u^\epsilon$  converges in  $L^2(D)$  to the limit  $u \in W^{1,2}(D)$  which solves the boundary value problem*

$$\begin{aligned} \Delta u - (\lambda + c)u &= f, \quad x \in D \\ u &= 0, \quad x \in \partial D \end{aligned} \tag{5.1.5}$$

*Proof.* Note that the (extended by zero) solution  $u^\epsilon$  of (5.1.1) is the minimizer in  $D$  of the functional  $\Gamma^\epsilon[u^\epsilon] = \int_{G^\epsilon(\omega) \cap D} |\nabla u^\epsilon|^2 + \lambda |u^\epsilon|^2 + 2fu^\epsilon dx = \int_D |\nabla u^\epsilon|^2 + \lambda |u^\epsilon|^2 + 2fu^\epsilon dx$  over the

class of functions  $u^\epsilon \in W^{1,2}(G^\epsilon(\omega))$ .

Thus,  $\Gamma^\epsilon[u^\epsilon] \leq \Gamma^\epsilon[0] = 0$  which implies that  $\int_{G^\epsilon(\omega) \cap D} |\nabla u^\epsilon|^2 + \lambda |u^\epsilon|^2 dx \leq 2\|u^\epsilon\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)}$ . Using the Friedrichs inequality  $\|u^\epsilon\|_{L^2(D)} \leq C\|\nabla u^\epsilon\|_{L^2(D)}$ , we obtain that  $\|u^\epsilon\|_{L^2(D)} \leq C$ , where  $C$  is independent of  $\epsilon$ . Since  $u^\epsilon$  is bounded, it has a subsequence, still denoted by  $u^\epsilon$ , that converges weakly in  $W^{1,2}(D)$  and strongly in  $L^2(D)$  to some function  $u \in W^{1,2}(D)$ .

To prove the theorem, it is enough to show that the limit  $u$  is the minimizer of the functional

$$\tilde{\Gamma}[u] = \int_D |\nabla u|^2 + (\lambda + c) |u|^2 + 2fudx.$$

Step1: We first establish the inequality  $\limsup_{\epsilon \rightarrow 0} \Gamma^\epsilon[u^\epsilon] \leq \tilde{\Gamma}[w]$  for all  $w \in W^{1,2}(D)$ .

For this purpose, we consider a partition of  $D$  with cubes  $Q^\alpha = Q(x^\alpha, h)$  centered at  $x^\alpha$  of size  $h$ , so that  $\cup_\alpha Q(x^\alpha, h)$  is a cover of  $D$  and the points  $x^\alpha$  form a periodic lattice of period  $h - r$ ,  $r$  to be chosen. Consider a partition of unity  $\{\phi_\alpha\}$  of  $C^2$  functions such that

1.  $0 \leq \phi_\alpha \leq 1$
2.  $\phi_\alpha = 0$  if  $x \notin Q^\alpha$ ,  $\phi_\alpha = 1$  if  $x \in Q^\alpha \setminus \cup_{\beta \neq \alpha} Q^\beta$
3.  $\sum_\alpha \phi_\alpha(x) = 1$ , if  $x \in D$
4.  $|\nabla \phi_\alpha| \leq C/r$

Let us denote by  $v^\alpha = v^{\alpha(\epsilon)}$  the minimizer of (5.1.4) in the cube centered at  $x^\alpha$ . For  $w \in C^2(D)$ , compactly supported in  $D$ , define

$$w_h^\epsilon(x) = \sum_{\alpha=1}^{n(h)} w(x) v^\alpha(x) \phi_\alpha(x) = w(x) + \sum_{\alpha=1}^{n(h)} w(x) [v^\alpha(x) - 1] \phi_\alpha(x) \quad (5.1.6)$$

so that  $w_h^\epsilon \in W^{1,2}(D)$  and  $w_h^\epsilon(x) = 0$  in  $F^\epsilon(\omega)$ . Thus,

$\Gamma^\epsilon[u^\epsilon] \leq \Gamma^\epsilon[w_h^\epsilon]$ . Under our assumptions,

$$\int_{Q_h^\alpha} |\nabla v^\alpha|^2 + h^{-2-\gamma} |v^\alpha - 1|^2 dx \leq Ch^n \quad (5.1.7)$$

which implies that

$$\int_{Q_h^\alpha} |\nabla v^\alpha|^2 dx \leq Ch^n \quad (5.1.8)$$

and

$$\int_{Q_h^\alpha} |v^\alpha - 1|^2 dx \leq Ch^{n+\gamma+2} \quad (5.1.9)$$

We denote by  $\hat{Q}_h^\alpha = Q_h^\alpha \setminus \cup_{\beta \neq \alpha} Q_h^\beta$  the concentric cube centered at  $x^\alpha$  of size  $\hat{h} = h - 2r$ . Then,

$$\begin{aligned} & \int_{Q_h^\alpha \setminus \hat{Q}_h^\alpha} |\nabla v^\alpha|^2 dx + h^{-2-\gamma} \int_{Q_h^\alpha \setminus \hat{Q}_h^\alpha} |v^\alpha - 1|^2 dx \\ &= \int_{Q_h^\alpha} |\nabla v^\alpha|^2 + h^{-2-\gamma} |v^\alpha - 1|^2 dx - \int_{\hat{Q}_h^\alpha} |\nabla v^\alpha|^2 + h^{-2-\gamma} |v^\alpha - 1|^2 dx + O(rh^{n-1}) \\ &\leq \text{cap}(x, h, \epsilon, \gamma, \omega) - \text{cap}(x, \hat{h}, \epsilon, \gamma, \omega) + O(rh^{n-1}) \end{aligned}$$

.

Choosing  $r = h^{1+\gamma/2} = O(h)$  we obtain straightforward that

$$\int_{Q_h^\alpha \setminus \hat{Q}_h^\alpha} |v^\alpha - 1|^2 dx = O(h^{n+\gamma+2}) \quad (5.1.10)$$

and

$$\int_{Q_h^\alpha \setminus \hat{Q}_h^\alpha} |\nabla v^\alpha|^2 dx = O(h^n) \quad (5.1.11)$$

Differentiating (5.1.4), we have

$$\frac{\partial w_h^\epsilon}{\partial x_i} = \frac{\partial w}{\partial x_i} + \sum_\alpha \frac{\partial w}{\partial x_i} (v^\alpha - 1) \phi_\alpha + \sum_\alpha \frac{\partial v^\alpha}{\partial x_i} w \phi_\alpha + \sum_\alpha \frac{\partial \phi_\alpha}{\partial x_i} (v^\alpha - 1) w \quad (5.1.12)$$

We substitute (5.1.10) into  $\Gamma^\epsilon[w_h^\epsilon]$  to obtain

$$\begin{aligned} \Gamma^\epsilon[w_h^\epsilon] &= \int_{G^\epsilon(\omega)} |\nabla w_h^\epsilon|^2 + \lambda |w_h^\epsilon|^2 + 2f w_h^\epsilon dx \\ &= \int_{G^\epsilon(\omega)} |\nabla w|^2 + \lambda |w|^2 + 2f w dx + \sum_{\alpha=1}^{n(h)} \int_{Q_h^\alpha} |\nabla v^\alpha|^2 w^2 \phi_\alpha^2 dx + \sum_{i=1}^5 L_i(\epsilon, h) \end{aligned}$$

where

$$\begin{aligned} L_1(\epsilon, h) &= \sum_{\alpha}^{N(h)} 2 \int_{Q_h^\alpha} (f + \lambda w)(v^\alpha - 1) w \phi_\alpha dx \\ L_2(\epsilon, h) &= \sum_{\alpha, \beta}^{N(h)} 2 \int_{Q_h^\alpha \cap Q_h^\beta} \left\{ \sum_{i=1}^n \left( \frac{\partial w}{\partial x_i} \phi_\alpha + \frac{\partial \phi_\alpha}{\partial x_i} w \right) \left( \frac{\partial w}{\partial x_i} \phi_\beta + \frac{\partial \phi_\beta}{\partial x_i} w \right) \right. \\ &\quad \left. + \lambda w^2 \phi_\alpha \phi_\beta \right\} (v^\alpha - 1)(v^\beta - 1) dx \end{aligned}$$

$$L_3(\epsilon, h) = \sum_{\alpha, \beta}^{N(h)} 2 \int_{Q_h^\alpha \cap Q_h^\beta} \sum_{i=1}^n \left( \frac{\partial w}{\partial x_i} \phi_\alpha + \frac{\partial \phi_\alpha}{\partial x_i} w \right) \left( \frac{\partial w}{\partial x_i} \phi_\beta + \frac{\partial \phi_\beta}{\partial x_i} w \right) (v^\beta - 1) dx$$

$$L_4(\epsilon, h) = \sum_{\alpha, \beta}^{N(h)} \sum_{i=1}^n 2 \int_{Q_h^\alpha \cap Q_h^\beta} \frac{\partial v_h^\alpha}{\partial x_i} \frac{\partial v_h^\beta}{\partial x_i} \phi_\alpha \phi_\beta w^2 dx$$

$$L_5(\epsilon, h) = \sum_{\alpha, \beta}^{N(h)} \sum_{i=1}^n 2 \int_{Q_h^\alpha} \frac{\partial w}{\partial x_i} w \phi_\alpha \frac{\partial (v_h^\alpha - 1)}{\partial x_i} dx$$

Taking into account the properties of  $\phi_\alpha$ , (5.1.7) – (5.1.9) and the fact that the number of cubes  $Q_h^\beta$  which intersect  $Q_h^\alpha$  is no more than  $3^n$ , we have

$$\lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \sum_{i=1}^5 L_i(\epsilon, h) = 0$$

From the properties of smooth functions  $\phi_\alpha$  we now have

$$\int_{Q_h^\alpha} w^2 \phi_\alpha |\nabla v^\alpha|^2 dx \leq \int_{Q_h^\alpha} w^2 [|\nabla v^\alpha|^2 + h^{-2-\gamma} |v^\alpha - 1|^2] dx \leq \bar{w}_\alpha^2 \text{cap}(x^\alpha, h, \epsilon, \gamma, \omega), \text{ where } \bar{w}_\alpha \text{ is the mean of } w_\alpha \text{ over } Q_h^\alpha.$$

Summing over the cubes and letting  $\epsilon$  tend to zero we have

$$\limsup_{\epsilon \rightarrow 0} \sum_{\alpha} \int_{Q_h^\alpha} |\nabla v^\alpha|^2 w^2 \phi_\alpha^2 dx \leq \int_D c w^2 dx + O(h) \quad (5.1.13)$$

Combine these inequalities to see that  $\Gamma^\epsilon[u^\epsilon] \leq \bar{\Gamma}[w]$  for all twice differentiable functions with compact support in  $D$ . Using a density argument, this inequality holds for all  $w \in H_0^1(D)$ .

Step 2: To show the reverse inequality  $\liminf_{\epsilon \rightarrow 0} \Gamma^\epsilon[u^\epsilon] \geq \bar{\Gamma}[u]$ , pick a sequence  $u_\delta(x) \in C_0^1(D)$  such that  $\|u_\delta - u\|_{H_0^1(D)} \leq \epsilon$ , where  $u$  is the weak limit of the sequence of minimizers  $u^\epsilon$  of  $\Gamma^\epsilon[\cdot]$  in  $H_0^1(D)$ .

According to lemma 3.2 on [12] pg.73, there is a sequence  $\{u_\delta^\epsilon\} \in H_0^1(D, F^\epsilon(\omega)) = \{v \in H_0^1(D) : v = 0 \text{ in } F^\epsilon(\omega)\}$  that converges to  $u_\delta$  and satisfies  $\|u_\delta^\epsilon - u^\epsilon\|_{H^1(D)} \leq C\|u_\delta - u\|_{H^1(D)}$ .

Take now the cubes  $Q_h^\alpha$  that belong to the set  $D_{\tilde{\delta}} = \{x \in D : |u_\delta(x)| \geq \tilde{\delta}\}$  for positive parameter  $\tilde{\delta}$  and in each of these cubes define the function  $v_\alpha^\epsilon = \frac{u_\delta^\epsilon}{u^\epsilon}$  so that  $v_\alpha^\epsilon \rightarrow 1$  weakly in  $L^2(Q_h^\alpha)$ .

Clearly,

$$\int_{Q_h^\alpha} |\nabla v_\alpha^\epsilon|^2 + h^{-2-\gamma} |v_\alpha^\epsilon - 1|^2 dx \geq \text{cap}(x^\alpha, \epsilon, h, \gamma, \omega) \quad (5.1.14)$$



and

$$\frac{\partial v_\alpha^\epsilon}{\partial x_i} = \frac{1}{u_\delta} \frac{\partial u_\delta^\epsilon}{\partial x_i} - \frac{1}{u_\delta} \frac{\partial u_\delta}{\partial x_i} - \frac{u_\delta^\epsilon - u_\delta}{u_\delta^2} \frac{\partial u_\delta}{\partial x_i} \quad (5.1.15)$$

Using the expansion (5.1.13) in (5.1.12) and taking into account that  $u_\delta^\epsilon \rightarrow u_\delta$  strongly, we get

$$\int_{Q_h^\alpha} |\nabla u_\delta^\epsilon|^2 dx \geq \text{cap}(x^\alpha, \epsilon, h, \gamma, \omega) [\min_{Q_h^\alpha} |u_\delta|]^2 + \frac{[\min_{Q_h^\alpha} |u_\delta|]^2}{[\max_{Q_h^\alpha} |u_\delta|]^2} \int_{Q_h^\alpha} |\nabla u_\delta|^2 dx - G(\epsilon, \delta, \tilde{\delta}, h) \quad (5.1.16)$$

where  $\lim_{\epsilon \rightarrow 0} G(\epsilon, \delta, \tilde{\delta}, h) = 0$  for fixed  $h, \delta, \tilde{\delta}$ .

Finally, we sum over all cubes that intersect  $G^\epsilon(\omega)$ ,

$$\begin{aligned} \Gamma^\epsilon[u_\delta^\epsilon] &\geq \sum_{\alpha=1}^N \int_{Q_h^\alpha} |\nabla u_\delta|^2 + \sum_{\alpha=1}^N \frac{\text{cap}(x^\alpha, \epsilon, h, \omega)}{h^n} [\sup_{Q_h^\alpha} |u_\delta|]^2 h^n + \int_D \lambda |u_\delta^\epsilon|^2 + 2f u_\delta^\epsilon \\ &\quad - \sum_{i=1}^n \frac{[\sup_{Q_h^\alpha} |u_\delta|]^2 - [\min_{Q_h^\alpha} |u_\delta|]^2}{[\sup_{Q_h^\alpha} |u_\delta|]^2} \int_{Q_h^\alpha} |\nabla u_\epsilon|^2 dx - NG(\epsilon, \delta, \tilde{\delta}, h) \end{aligned}$$

We let  $h \rightarrow 0$  for fixed  $\delta$  to obtain

$$\lim_{\epsilon \rightarrow 0} \Gamma^\epsilon[u_\delta^\epsilon] \geq \int_{D_\delta} |\nabla u_\delta|^2 + c u_\delta^2 dx + \int_D \lambda u_\delta^2 + 2f u_\delta^2 dx$$

Let now  $\delta, \tilde{\delta}, \epsilon$  tend to zero:  $\bar{\Gamma}[u] \leq \bar{\Gamma}[v]$  for all  $v \in H_0^1(D)$ . □

### 5.1.3 Boolean models and the strange term for biharmonic operators

Our goal in this section is to show that the extension of the result in [7] and [6] is a consequence of appropriate random modeling. As we described in section 4.2.2 the modeling of a domain perforated randomly by balls of random radius can be obtained from the use of Boolean models. For the purpose of homogenization problems, the difficulty arises if we allow the balls to intersect or be tangential to each other. Thus, we can either restrict the realization of the Boolean models or modify appropriately as before the RCM model:

Suppose the RCM  $(X, g, \lambda)$  is applied on a bounded region  $D$  of  $\mathbb{R}^n$ . We want to assign every endpoint of the line process as the center of a ball of random radius. For fixed  $\epsilon > 0$  there is a set of points from the point process  $X$ . We let every point be the center of a ball with random radius  $\rho(\omega) \leq \min d(x_i, x_j)(\omega)$ , where  $\rho(\omega)$  is indentially distributed random variable  $s(x, \omega)$  taking maximum value  $\min d(x_i, x_j)(\omega)$ :

The ergodic properties of the structure imply that the capacity-type functionals have stationary properties. In particular, let us denote by  $B_{\rho_\epsilon}(x, \omega)$  be the random ball centered at  $x \in X$  with random radius  $\rho_\epsilon(\omega)$ . We denote by  $F^\epsilon(\omega) = \bigcup_{x \in X} B_{\rho_\epsilon}(x, \omega)$  and  $G^\epsilon(\omega) = D \setminus F^\epsilon(\omega)$ .

We consider the Dirichlet problem

$$\begin{aligned} -\Delta^2 u^\epsilon &= f \quad \text{in } G^\epsilon(\omega) \\ u^\epsilon &\in H_0^2(G^\epsilon(\omega)) \end{aligned} \tag{5.1.17}$$

that corresponds to the obstacle problem

$$\min \left\{ \int_{\Omega} \frac{1}{2} (\Delta u)^2 - f u, u \geq 0 \text{ in } G^\epsilon(\omega) \right\} \tag{5.1.18}$$

for  $f \in L^2(\Omega)$ .

Elliptic theory guarantees the existence of the solutions to (5.1.15). We extend the solutions by zero to the holes and we keep the same notation for the functions

$$\tilde{u}^\epsilon \equiv u^\epsilon = \begin{cases} u^\epsilon & \text{in } G^\epsilon(\omega) \\ 0 & \text{in } F^\epsilon(\omega) \end{cases}$$

In the weak formulation  $\int_{G^\epsilon(\omega)} \Delta u^\epsilon \Delta v dx = \int_{G^\epsilon(\omega)} f v dx$ ,  $v \in H_0^2(G^\epsilon(\omega))$ , we choose  $v = u^\epsilon$  to obtain

$$\|\Delta u^\epsilon\|_{L^2(G^\epsilon(\omega))}^2 \leq \|f\|_{L^2(D)} \|u^\epsilon\|_{L^2(G^\epsilon(\omega))} \tag{5.1.19}$$

Since  $\int_{G^\epsilon(\omega)} \|\nabla u^\epsilon\|^2 dx = - \int_{G^\epsilon(\omega)} u^\epsilon \Delta u^\epsilon dx$ , it follows that

$$\|\nabla u^\epsilon\|_{L^2(G^\epsilon(\omega))}^4 \leq \|u^\epsilon\|_{L^2(G^\epsilon(\omega))}^3 \|f\|_{L^2(D)}$$

Finally, Poincaré's inequality gives  $\|u^\epsilon\|_{H^2(G^\epsilon(\omega) \cap D)} \leq c(\Omega) \|f\|_{L^2(D)}$ . It follows that the sequence  $\{u^\epsilon\}$  is bounded in  $H_0^2(D)$ . Consequently, up to a subsequence,  $u^\epsilon$  has a weak limit  $u \in H_0^2(D)$ .

Under the construction of an appropriate corrector  $\mu \in W^{-1,\infty}(D)$  we expect the limit function  $u$  to satisfy the Dirichlet problem

$$\begin{aligned} -\Delta^2 u + \mu u &= f \quad \text{in } D \\ u &\in H_0^2(D) \end{aligned} \tag{5.1.20}$$

In particular, we assume for the moment that the following assumptions hold true:

- there is a sequence  $w^\epsilon \in H_0^2(D)$  that  $w^\epsilon = 0$  in each  $B_{\rho_\epsilon}(x, \omega)$
- for every sequence  $v^\epsilon$  satisfying  $v^\epsilon = 0$  in each  $B_{\rho_\epsilon}(x, \omega)$  and  $v^\epsilon \rightharpoonup v$  weakly in  $H^1(D)$ , we have  $\langle -\Delta^2 w^\epsilon, \phi v^\epsilon \rangle_{H^{-2}(D), H_0^2(D)} \rightarrow \langle \mu, \phi v \rangle_{H^{-2}(D), H_0^2(D)}$  for all  $\phi \in C_0^\infty(D)$ .
- $w^\epsilon = 1$ ,  $\frac{\partial w^\epsilon}{\partial \eta} = 0$  in  $F^\epsilon(\omega)$  and
- $w^\epsilon = 0$ ,  $\frac{\partial w^\epsilon}{\partial \eta} = 0$  on  $\partial D \setminus F^\epsilon(\omega)$
- $w^\epsilon \rightarrow 1$  weak  $H^2$

Under the previous assumptions we obtain the following theorem:

**Theorem 5.1.3.** *The sequence of solutions  $u^\epsilon$  converges weakly in  $H_0^2(D)$  to the unique solution of (5.1.16), that is, the minimizer of  $J[u] = \int_D \frac{1}{2}(\Delta u)^2 + \frac{1}{2}\mu u^2 - f u dx$  over  $H_0^2(D)$ .*

*Proof.* Note that, for  $\phi \in C_0^\infty(D)$ , the function  $v = w^\epsilon \phi$  is in  $H_0^2(G^\epsilon(\omega))$ . Choosing this as test function in the weak formula,

$$\int_{G^\epsilon(\omega)} \Delta u^\epsilon \Delta(w^\epsilon \phi) dx = \int_{G^\epsilon(\omega)} f w^\epsilon \phi dx. \quad (5.1.21)$$

Since  $\Delta(w^\epsilon \phi) = \phi \Delta w^\epsilon + w^\epsilon \Delta \phi + 2 \nabla \phi \nabla w^\epsilon$ , (5.1.19) becomes

$$\begin{aligned} 2 \int_{G^\epsilon(\omega)} \Delta u^\epsilon \nabla \phi \nabla w^\epsilon dx + \int_{G^\epsilon(\omega)} \phi \Delta u^\epsilon \Delta w^\epsilon dx + \int_{G^\epsilon(\omega)} w^\epsilon (\Delta u^\epsilon \Delta \phi) dx &= \int_{G^\epsilon(\omega)} f w^\epsilon \phi dx, \text{ or} \\ \int_{G^\epsilon(\omega)} w^\epsilon (\Delta u^\epsilon \Delta \phi) dx + \int_{G^\epsilon(\omega)} \Delta(u^\epsilon \phi) \cdot \Delta w^\epsilon dx &= \int_{G^\epsilon(\omega)} f w^\epsilon \phi dx + T \end{aligned} \quad (5.1.22)$$

where  $T = \int_{G^\epsilon(\omega)} u^\epsilon \Delta w^\epsilon \cdot \Delta \phi + 2 \Delta w^\epsilon \nabla u^\epsilon \cdot \nabla \phi - 2 \Delta u^\epsilon \nabla \phi \nabla w^\epsilon dx$

Since  $w^\epsilon \rightarrow 1$  in  $H^2$  it follows that  $T \rightarrow 0$

$$\begin{aligned} \int_{G^\epsilon(\omega)} w^\epsilon (\Delta u^\epsilon \Delta \phi) dx &\rightarrow \int_D \Delta u \Delta \phi dx \text{ and} \\ \int_{G^\epsilon(\omega)} f w^\epsilon \phi dx &\rightarrow \int_D f \phi dx. \end{aligned}$$

For the second term of the left side of (5.1.20), we recall Green's identity

$$\int_B \Delta u \Delta v dx = \int_{\partial B} \Delta u \frac{\partial v}{\partial \eta} ds - \int_{\partial B} \frac{\partial \Delta u}{\partial \eta} v ds.$$

Then, since  $\int_D \phi \Delta^2 w^\epsilon u^\epsilon dx \rightarrow (\mu, \phi u)$ , the result follows.  $\square$

Using the previous hypothesis on the corrector, we can also prove the following

**Proposition 5.1.4.** *If the above hypothesis is satisfied,  $\langle \mu, \phi \rangle = \lim_{\epsilon \rightarrow 0} \int_D |\Delta w^\epsilon|^2 \phi dx$ .*

*Also  $\liminf_{\epsilon \rightarrow 0} J[u^\epsilon] \geq J[u]$  whenever  $u^\epsilon \rightharpoonup u$  weakly in  $H^2$ .*

Please see [6] for the corresponding results in the case of Poisson problem. Our main consideration is to construct the corrector  $\mu$  and the function  $w^\epsilon$ . An important tool for this construction is the fundamental solution of the biharmonic equation

$h(x) = \frac{1}{2(4-n)(2-n)w_{n-1}} \frac{1}{x^{n-4}}$ . We consider only the case  $n > 4$ . Let  $a^\epsilon(r) = r\epsilon^{n/n-4}$ . Then,

$\text{cap} B_r = 2(4-n)(2-n)w_{n-1}r^{n-4}$ . Note that, thanks to the properties of our model, we can assume that  $\text{cap}(B_{\rho_\epsilon}(x, \omega)) = \epsilon^n s(x, \omega)$  and

$$\rho(x, \omega) = \left( \frac{s(x, \omega)}{2(4-n)(2-n)w_{n-1}} \right)^{1/n-4}$$

The rescaled corrector  $w^\epsilon$  that we need satisfies  $w^\epsilon = \epsilon^4 v(x/\epsilon, \omega)$ , where  $v(x/\epsilon, \omega)$  satisfies

$$\begin{aligned} -\Delta^2 v &= \mu, \quad x \in \epsilon^{-1} G^\epsilon(\omega) \\ v &= 1/\epsilon^4, \quad x \in F^\epsilon(\omega) \end{aligned} \tag{5.1.23}$$

If  $h_i$  is the fundamental solution in the ball centered at  $x_i \in X$ , that is

$\Delta^2 h_i = -s(x_i, \omega) \delta(x - x_i)$ , we look for  $v(x/\epsilon, \omega)$  that solves the Dirichlet problem

$$\begin{aligned} \Delta^2 v &= \mu - \sum_{x_i \in X} s(x_i, \omega) \delta(x - x_i), \quad x \in D \\ v &= 0, \quad \frac{\partial v}{\partial \eta} = 0 \quad x \in \partial D \end{aligned} \tag{5.1.24}$$

Consider now the obstacle problem

$\bar{v}_{\mu, A}(x, \omega) = \inf \{v : \Delta v^2 \leq \mu - \sum_{\{x_i \in X\} \cap A} s(x_i, \omega) \delta(x - x_i), v \geq 0 \text{ in } A, v = 0 \text{ on } \partial A\}$ . Note that

the function  $h_{\mu, x} = \frac{\mu}{24n^2} |x - x_i|^4 + h_i(x - x_i)$  also satisfies (5.1.22). Using maximum principle,

$$\bar{v}_{\mu, A}(x, \omega) \geq h_{\mu, x} - \frac{1}{24n^2} \mu - r^{n-4}$$

where  $h_{\mu, x} = \frac{\mu}{24n^2} |x - x_i|^2 + h_i(x - x_i)$ , for  $n \geq 5$ .

The contact set is defined by  $\bar{m}_\mu(A, \omega) = |\{x \in A : \bar{v}_{\mu, A}(x, \omega) = 0\}|$ . Due to our assumptions, this set has ergodic properties. The ergodic theorem then implies the existence of the limit

$$\lim_{\epsilon \rightarrow 0} \frac{\left| \{y; \epsilon^4 \bar{v}_{\mu, B_{\epsilon^{-1}}(\epsilon^{-1}x_i)}(x, \omega)\} \right|}{|B_1(0)|} = \lambda(\mu)$$

and the limit is positive as long as the random capacity remains in a critical range.

In particular, one can show that  $\lambda(\mu)$  satisfies the following properties:

- $\lambda(\mu)$  is nondecreasing function on  $\mu$
- for all  $\mu$ ,  $\lambda(\mu) \geq 0$
- for  $\mu \geq c(n) \sup \rho(x, \omega)^{n-4}$ ,  $\lambda(\mu) \geq 0$

Finally, if we define  $\bar{\mu} = \sup\{\alpha; \lambda(\mu) = 0\}$ , then  $\bar{\mu} \geq 0$  and the corrector  $w^\epsilon(x, \omega) = \inf\{w : \Delta^2 w \leq \bar{\mu}, w \geq 1 \text{ in } G^\epsilon(\omega), w = 0 \text{ on } \partial\Omega \setminus F^\epsilon(\omega)\}$  satisfies the needed properties.

The remaining part of the proof follows the lines of [6].

## 5.2 The Neumann problem

Neumann problems can be treated in a similar manner with Dirichlet problems, as long as we have a good notion of compactness of the solutions  $u^\epsilon$ . The difference is that extension by zero is not valid in this case. The notion of strongly connected domains and its variations, due to Khrušlov, is a quite general and powerful method which allows the extension of function in  $W^{1,2}(\Omega)$ .

Here, we consider elliptic equations of the form

$$\begin{aligned} -\Delta u^\epsilon + \lambda u^\epsilon &= f^\epsilon, \quad x \in G^\epsilon(\omega) \\ \frac{\partial u^\epsilon}{\partial \eta} &= 0, \quad x \in \partial G^\epsilon(\omega) \\ u^\epsilon &= 0, \quad x \in \partial D \end{aligned} \tag{5.2.1}$$

for  $u^\epsilon \in W^{1,2}(G^\epsilon(\omega))$ ,  $f^\epsilon \in L^2(G^\epsilon(\omega))$  and as  $\epsilon \rightarrow 0$ ,  $G^\epsilon(\omega)$  becomes denser in  $D$ . As before, the set  $G^\epsilon(\omega) = D \setminus F^\epsilon(\omega)$  is the complement set (vacant region) of the realization of the random connection model.

To deal with these problems, we will further assume the following conditions hold true for  $F^\epsilon(\omega)$ :

- For any  $\epsilon > 0$ , there is a finite number of points  $\{y_i^\epsilon\}$ ,  $i = 1, \dots, n(\epsilon)$  such that the balls  $B(y_i^\epsilon, \mu\rho_\epsilon) = B_i^\epsilon$  is a finite cover of  $D$  for some constant  $\mu > 0$  independent of  $\epsilon$ ,  $B(y_i^\epsilon, \mu\rho_\epsilon) \cap F^\epsilon(\omega)$  and  $B(y_i^\epsilon, \mu\rho_\epsilon) \setminus F^\epsilon(\omega)$  are bi-Lipschitz isomorphic to either (i) circular cylinder and annular cylinder (respectively), or (ii) star-shaped domain centered at the origin and its complement in a bigger ball. That is, there exist bi-Lipschitz isomorphism  $G^{n(\epsilon)} : B^\epsilon \rightarrow G^{n(\epsilon)}(B^\epsilon)$  so that  $G^{n(\epsilon)}(l_{ij}) \subset \{x \in \mathbb{R}^n : x_1 = x_2 = \dots = x_{n-1} = 0\}$  and for any  $x_1, x_2 \in T_{\rho_\epsilon}(l_{ij})$  there is constant  $C > 0$  independent of  $\epsilon$  such that  $\frac{1}{C} |x_1 - x_2| \leq |G^\epsilon(x_1) - G^\epsilon(x_2)| \leq C |x_1 - x_2|$  and  $G^{n(\epsilon)}(B^\epsilon \cap G^\epsilon(\omega)) = (T_{2\rho_\epsilon}(l_{ij}) \setminus T_{\rho_\epsilon}(l_{ij}))$ .  
or  $G^{n(\epsilon)}(B^\epsilon \cap F^\epsilon(\omega)) = J^\epsilon$  is a star shaped domain centered at the origin with Lipschitz boundary in the ball  $B_{4\epsilon}(0)$ .
- There is a maximal number  $M > 0$  such that at most  $M$  sets  $\{T_{\rho_\epsilon}(l_{ij}), i, j = 1, \dots, n(\epsilon), i \neq j\}$  and at most  $M$  sets  $\{B(y_i^\epsilon, \mu\rho_\epsilon)\}$  with common points exist.

The last assumptions do not hold for every random model, unless, for instance we choose sufficiently small radius. For example, two cylindrical channels can be parallel with tangential boundaries.

We call such domains strongly connectable. To give a concrete case for which these assumptions hold, let us consider the following discretized models: we take the  $\epsilon Z^3$  lattice at which two neighbor-points have distance  $\epsilon$ : for each plane  $z = z_i$ ,  $i = 1, \dots, n(\epsilon)$  with  $|z_{i+1} - z_i| = \epsilon$  we take a marked point process on the  $Z^3$  lattice,  $X$ . That is, a point process that randomly counts point the lattice. For the points  $x, y$  which belong to the process, we take the discrete connection function

$$g(|x - y|) = \begin{cases} p & \text{if } x, y \in X \text{ and } \epsilon \leq |x - y| \leq \sqrt{2}\epsilon \\ 0 & \text{otherwise} \end{cases}$$

This connection function allows connectivity of neighboring points as well as diagonal connec-

tions between the planes  $z_i, z_{i+1}$ . For  $\rho \ll \epsilon$ , we assign a tube of radius  $\rho$  for each open line segment.

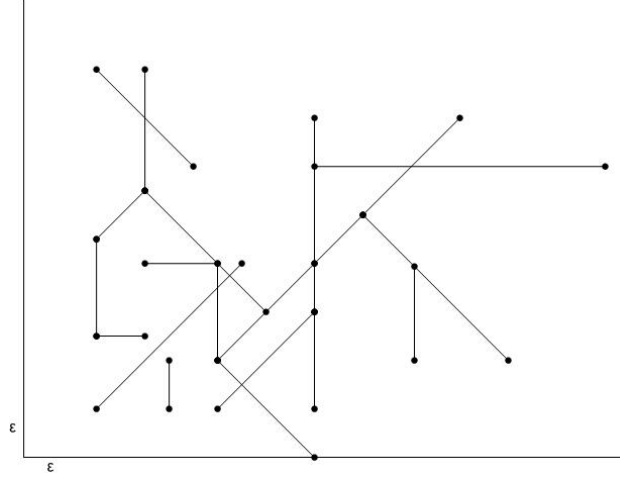


Figure 5.1 Discretization of a random connection model

In the three dimensional  $\epsilon\mathbb{Z}^3$  lattice, every point on the lattice has 14 neighboring points with distance either  $\epsilon$  or  $\sqrt{2}\epsilon$ . The total volume which can be obtained from all connections does not exceed  $4(\frac{4}{3}\pi\rho^3 + \pi\rho^2\epsilon) + 10(\frac{4}{3}\pi\rho^3 + \pi\rho^2\sqrt{2}\epsilon)$ . Since the total volume of this region is  $8\epsilon^3$ , if we choose for instance  $\rho = \epsilon/20$ , we guarantee the existence of a two phase domain. According to this example, and due to symmetry of the possible random structures, the domain  $F^\epsilon(\omega)$  consists of two types of domains:

1.  $L^\epsilon(\omega) = \cup \tilde{T}_{\rho_\epsilon}(l_{ij})$  is the set of tubules such that for each tubule  $\tilde{T}_{\rho_\epsilon}(l_{ij}) \subset T_{\rho_\epsilon}(l_{ij})$ ,  $\tilde{T}_{2\rho_\epsilon}(l_{ij}) \setminus \tilde{T}_{\rho_\epsilon}(l_{ij}) \in G^\epsilon(\omega)$ , and
2.  $J^\epsilon(\omega) = \cup \left( T_{\rho_\epsilon}(l_{ij}) \setminus \tilde{T}_{\rho_\epsilon}(l_{ij}) \right) = F^\epsilon(\omega) \setminus L^\epsilon(\omega)$  is the remaining domain that includes the set of intersections among the tubules. For instance, consider a cubic lattice of length and width of size  $\epsilon$  and height  $\sqrt{\epsilon}$  as in Figure 5.1.

Suppose that all line segments are open so that we have a region of with all possible tubes. There are 4 diagonals that pass through the center of the cube, 4 diagonals which connect the upper with the lower corners and the tubes on each parallel plane of the quadrilateral. It is easy to see that under our setting, each intersection is a star-shaped

domain.

The same properties hold if we allow our connection function to be 1 for more discrete values, i.e.  $g(|x - y|) = 1$  for  $|\cdot| \in \{\epsilon, \dots, c\epsilon\}$ .

### 5.2.1 Local Characteristics

To pass the limit to the equation, we first define the following quantities:

1. The volume capacity

$$\beta(z, h, \epsilon, \omega) = \int_{G^\epsilon(\omega) \cap Q_h^z} \chi_{G^\epsilon(\omega)} dx = \text{mes}(G^\epsilon(\omega) \cap Q_h^z)$$

2. The functional

$$P_{\epsilon, h}^z(\xi) = \inf_{v^\epsilon \in W^{1,2}(Q_h^z \cap G^\epsilon(\omega))} \int \{|\nabla v^\epsilon|^2 + h^{-2-\gamma} |v^\epsilon - (x - z, \xi)|^2\} dx$$

Note that the function  $v_\xi^\epsilon$  which minimizes  $P_{\epsilon, h}^z(\xi)$  for any  $\xi \in \mathbb{R}^n$  can be written in the form  $v_\xi^\epsilon = \sum_{i=1}^n \xi_i v_i^\epsilon$ , where  $v_i^\epsilon$  is the corresponding minimizer for  $\xi_i = e_i$ .

Thus, if we write

$$a_{ij}(z, \epsilon, h, \omega) = \int \nabla v_i^\epsilon \cdot \nabla v_j^\epsilon + h^{-2-\gamma} [v_i^\epsilon - (x_i - \xi_i)][v_j^\epsilon - (x_j - \xi_j)] dx, \text{ then } P_{\epsilon, h}^z(\xi) \text{ takes the quadratic form } P_{\epsilon, h}^z(\xi) = \sum_{i,j=1}^n a_{ij}(z, \epsilon, h, \omega) \xi_i \xi_j.$$

The matrix  $[a_{ij}]$  is the local mean conductivity tensor of the medium at the point  $z$ .

It can also be written the form  $a_{ij}(x, \epsilon, h, \omega) = \int_{Q_h^x \cap G^\epsilon(\omega)} \frac{\partial v_i^\epsilon}{\partial x_j} dx = \int_{Q_h^x \cap G^\epsilon(\omega)} \frac{\partial v_j^\epsilon}{\partial x_i} dx$ , since the minimizer of  $P_{\epsilon, h}^z(\xi)$  solves the boundary problem

$$\begin{aligned} -\Delta u_\xi^\epsilon + h^{-2-\gamma} u_\xi^\epsilon &= h^{-2-\gamma} (x - z, \xi), \quad x \in Q_h^z \cap G^\epsilon(\omega) \\ \frac{\partial u^\epsilon}{\partial \eta} &= 0, \quad x \in \partial Q_h^z \cap G^\epsilon(\omega) \end{aligned} \tag{5.2.2}$$

The temperature distribution  $u^\epsilon$  of the porous medium is the minimizer of  $J_{G^\epsilon(\omega)}[u^\epsilon] = \int_{G^\epsilon(\omega)} |\nabla u^\epsilon|^2 dx$  over the class  $\{u^\epsilon \in W^{1,2}(G^\epsilon(\omega)) : u^\epsilon = u_0 \text{ on } \partial(D)\}$ . Suppose for the moment that  $u^\epsilon$  can be extended to  $\tilde{u}^\epsilon$  such that

$\|\tilde{u}^\epsilon\|_{W^{1,2}(D)} \leq C$  uniformly with respect to  $\epsilon$  (see section 4 and [16] for improvements). Then,



up to a subsequence  $\tilde{u}^\epsilon$  converges to a function  $u$  in  $L^2(D)$ , which is almost linear to every sufficiently small cube  $K_h^z$ , i.e.  $u(x) = u(z) + (x - z, \nabla u(z)) + o(h^2)$ . Then, for  $\epsilon$  large enough,

$$\int_{Q_h^z \cap G^\epsilon(\omega)} |v^\epsilon(x) - (x - z, \nabla u(z))|^2 dx = O(h^{n+4})$$

where  $v^\epsilon(x) = u^\epsilon(x) - u(z)$ . Thus, we can assume that the function  $v^\epsilon(x)$  is the minimizer of

$$J_h^\epsilon(v^\epsilon) = \int_{Q_h^z \cap G^\epsilon(\omega)} |\nabla v^\epsilon|^2 dx = O(h^n)$$

Under this consideration,

$$J_h^\epsilon(v^\epsilon) - P_{\epsilon,h}^z(\nabla u(z)) = o(h^n)$$

as  $h \rightarrow 0$ . This description shows the essential idea of the definition of the conductivity tensor.

### 5.2.2 Extension from $W^{1,2}(G^\epsilon(\omega))$ to $W^{1,2}(D)$

We extend  $u \in W^{1,2}(G^\epsilon(\omega))$  to  $\tilde{u} \in W^{1,2}(D)$  using exterior cylinder-extension, following [16]. The idea of the process is the following:

1. Extend  $u$  in the set of links, to concentric tubules of double radius.
2. Show that the extension to the set of joints is also valid. Note that, by assumption, the joints have distance  $C\epsilon$  between them and are connected through the links.

**Proposition 5.2.1.** *For  $u \in W^{1,2}(T_{2\rho \setminus \rho, \epsilon})$ , there is an extension  $\tilde{u} \in W^{1,2}(T_{2\rho, \epsilon})$  such that  $\tilde{u} = u$  in  $T_{\rho, \epsilon}$  and*

$$\|\tilde{u}\|_{L^2(T_{2\rho \setminus \rho, \epsilon})} \leq C\|u\|_{L^2(T_{\rho, \epsilon})}, \quad \|\nabla \tilde{u}\|_{L^2(T_{2\rho \setminus \rho, \epsilon})} \leq C\|\nabla u\|_{L^2(T_{\rho, \epsilon})}, \quad \text{where } C \text{ is independent of } \epsilon.$$

*Proof.* Consider the function  $u^\epsilon(x) = u(\epsilon x)$  defined on  $T_1 = \left\{x \in \mathbb{R}^3 : \frac{1}{4} \leq x_1^2 + x_2^2 \leq 1, 0 \leq x_3 \leq 1\right\}$ .

Let  $J_1$  be the concentric cylinder  $J_1 = \{x \in \mathbb{R}^3 : 0 \leq x_1^2 + x_2^2 \leq 1, 0 \leq x_3 \leq 1\}$  so that  $u^\epsilon \in W^{1,2}(J_1)$ . Then,

$$\int_{T_\epsilon} |u|^2 dx = \epsilon^3 \int_{T_1} |u^\epsilon|^2 dx \quad \text{and} \quad \int_{T_\epsilon} |\nabla u|^2 dx = \epsilon \int_{T_1} |\nabla u^\epsilon|^2 dx.$$

Let  $\bar{u}^\epsilon(x_3)$  be the 2-dimensional average  $\bar{u}^\epsilon(x_3) = \frac{1}{|B_1 \setminus B_{1/2}|} \int_{B_{1/4}} u^\epsilon dx_1 dx_2$ .

Let also  $w^\epsilon(x) = u^\epsilon(x) - \bar{u}^\epsilon(x_3)$ . Then  $w^\epsilon$  can be extended to  $\tilde{w}^\epsilon$  in  $J_1/T_1$  in the following sense:

$$\tilde{w}^\epsilon = \begin{cases} g(x_1, x_2)w^\epsilon(f(x_1, x_2)x_1, f(x_1, x_2)x_2, x_3) & \text{in } \frac{1}{16} \leq x_1^2 + x_2^2 \leq \frac{1}{4}, \\ 0 & \text{if } 0 \leq x_1^2 + x_2^2 \leq 1/16 \end{cases},$$

where  $g(x_1, x_2) = 4((x_1^2 + x_2^2)^{1/2} - \frac{1}{4})$  and  $f(x_1, x_2) = 7 - 12(x_1^2 + x_2^2)^{1/2}$ .

Then  $\tilde{w}^\epsilon \in W^{1,2}(J_1)$ . Using Poincare's inequality, we obtain the inequalities

$$\|\tilde{w}^\epsilon\|_{L^2(J_1)} \leq C\|w^\epsilon\|_{L^2(T_1)},$$

$$\|\nabla \tilde{w}^\epsilon\|_{L^2(J_1)} \leq C\|w^\epsilon\|_{W^{1,2}(T_1)} \leq C\|\nabla w^\epsilon\|_{L^2(T_1)} \text{ and}$$

$$\|\tilde{u}^\epsilon\|_{L^2(J_1)} \leq C\|u^\epsilon\|_{L^2(T_1)},$$

$$\|\nabla \tilde{u}^\epsilon\|_{L^2(J_1)} \leq C\|\nabla u^\epsilon\|_{L^2(T_1)}.$$

Scaling back with respect to  $\epsilon$  we obtain the needed inequalities.  $\square$

**Remark 5.2.2.** Same extension is true for  $u \in C^\alpha(G^\epsilon(\omega))$ . That is,  $\|\tilde{u}\|_{C^\alpha(D)} \leq C\|u\|_{C^\alpha(G^\epsilon(\omega))}$ .

**Corollary 5.2.3.** For any  $u \in W^{1,p}(B^\epsilon(\omega))$  with  $1 \leq p \leq \infty$ , there exists extension operator  $E(u) = \tilde{u} \in W^{1,p}(B_{2\rho(\epsilon)}(\omega))$  such that  $\tilde{u} = u$  in  $B_{\rho(\epsilon)}(\omega)$  and  $\|\tilde{u}\|_{L^2(B_{2\rho(\epsilon)}(\omega))} \leq C\|u\|_{L^2(B_{\rho(\epsilon)}(\omega))}$  and  $\|\nabla \tilde{u}\|_{L^2(B_{2\rho(\epsilon)}(\omega))} \leq C\|\nabla u\|_{L^2(B_{\rho(\epsilon)}(\omega))}$

Once the extension is developed for the set of links, we can immediately show extension for strongly connected domains, using the following proposition:

**Proposition 5.2.4.** Suppose  $J$  is a star shaped domains centered at the origin with Lipschitz boundary. The map  $(\rho, s) \rightarrow (f(s)\rho, s)$  from  $B_\epsilon$  to  $J_\epsilon = J(\frac{x}{\epsilon})$  is bilipschitz with coefficients independent of  $\epsilon$ .

*Proof.* See [16].  $\square$

**Theorem 5.2.5.** Suppose  $F^\epsilon(\omega)$  is the random set obtained from the realization of RCM models and suppose that  $F^\epsilon(\omega) = J^\epsilon(\omega) \cup L^\epsilon(\omega)$ , where  $L^\epsilon(\omega)$  is the set of links and  $J^\epsilon(\omega)$  is the set of joints. Let  $G^\epsilon(\omega) = D \setminus F^\epsilon(\omega)$  and suppose both  $F^\epsilon(\omega)$  and  $J^\epsilon(\omega)$  are simply connected. Then, for every  $u \in W^{1,2}(G^\epsilon(\omega))$ , there exists extension operator  $E(u) = \tilde{u} \in W^{1,2}(D)$  such that  $\|\tilde{u}^\epsilon\|_{L^2(D)} \leq C\|u^\epsilon\|_{L^2(G^\epsilon(\omega))}$ ,  $\|\nabla \tilde{u}^\epsilon\|_{L^2(D)} \leq C\|\nabla u^\epsilon\|_{L^2(G^\epsilon(\omega))}$ .

*Proof.* Let  $\hat{u} \in W^{1,2}(G^\epsilon(\omega) \cup L^\epsilon(\omega))$  be the extension of  $u \in W^{1,2}(G^\epsilon(\omega))$  to the set of links as in proposition (5.2.4). Since  $J^\epsilon(\omega)$  consists of joint sets that have distance at least  $C\epsilon$  between them, according to propositions 5.2.3 and 5.2.6, we can extend  $\hat{u}$  to  $\tilde{u} \in W^{1,2}(D)$  so that the  $L^2$  norms are bounded with constant independent of  $\epsilon$ .  $\square$

### 5.2.3 Passing the limit

Our main homogenization theorem is the following:

**Theorem 5.2.6.** *Suppose that the limits*

$$\lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{a_{ij}(z, \epsilon, h, \omega)}{h^3} = a_{ij}(x)$$

and

$$\lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{\beta(z, \epsilon, h, \omega)}{h^3} = \beta(x)$$

exist pointwise for all  $x \in D$ .

Then, the (extended) sequence of solutions  $u^\epsilon$  of (5.2.1) converges to the solution  $u$  of the boundary value problem

$$\begin{aligned} -\frac{1}{\beta(x)} \sum_{i,j=1}^3 \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \lambda u &= f, \quad x \in D \\ \frac{\partial u^\epsilon}{\partial \eta} &= 0 \quad x \in \partial D \end{aligned} \tag{5.2.3}$$

in the sense that  $\lim_{\epsilon \rightarrow 0} \int_{G^\epsilon(\omega)} u^\epsilon - u^2 dx = 0$ .

*Proof.* We start by noticing that the solution  $u^\epsilon$  of 5.2.1 is the minimizer of the functional  $J^\epsilon[u^\epsilon] = \int_{G^\epsilon(\omega)} |\nabla u^\epsilon|^2 + \lambda |u^\epsilon|^2 + 2f^\epsilon u^\epsilon dx$  in  $W^{1,2}(G^\epsilon(\omega))$ . Assume that  $f^\epsilon$  converges to  $f \in L^2(\Omega)$  in the sense that  $\lim_{\epsilon \rightarrow \infty} \|f^\epsilon - f\|_{L^\infty} = 0$ . To prove the theorem, it is enough to show that the solution  $u$  is the minimizer of

$$J[u] = \int_D \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \lambda \beta u + 2\beta f(x) u dx. \tag{5.2.4}$$

Multiply (5.2.1) by  $u^\epsilon$  and integrate by parts to obtain  $\|u^\epsilon\|_{W^{1,2}(G^\epsilon(\omega))} \leq C \|f^\epsilon\|_{L^2(G^\epsilon(\omega))}$ .

Let now  $\phi \in C_0^\infty(G^\epsilon(\omega))$ . Set  $l = \sup_{\partial G^\epsilon(\omega)} u^\epsilon$  and choose  $\phi = (u^\epsilon - k)_+$ , with  $k > l$  in the weak

formulation  $\int_{G^\epsilon(\omega)} \nabla u^\epsilon \cdot \nabla \phi dx + \int_{G^\epsilon(\omega)} \lambda u^\epsilon \phi dx = \int_{G^\epsilon(\omega)} f^\epsilon \phi dx$ :  
 $\int_{G^\epsilon(\omega)} |\nabla u^\epsilon|^2 dx + \int_{G^\epsilon(\omega)} \lambda u^\epsilon (u^\epsilon - k)_+ dx = \int_{G^\epsilon(\omega)} f^\epsilon (u^\epsilon - k)_+ dx$ . Using embedding theorem, we  
obtain  $\left( \int_{A(k)} |u^\epsilon|^2 dx \right)^{2/p} \leq C \int_{A(k)} |f^\epsilon (u^\epsilon - k)_+| dx$ , where  $A(k) = \{x \in G^\epsilon(\omega) : u^\epsilon(x) > k\}$ .  
The Holder inequality gives

$$\int_{A(k)} |f^\epsilon (u^\epsilon - k)_+| dx \leq \left( \int_{A(k)} |(u^\epsilon - k)_+|^p dx \right)^{1/p} \left( \int_{A(k)} |f^\epsilon|^q dx \right)^{1/q}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

Since we chose  $h > k$ , this implies that  $A(h) \subset A(k)$  and that  $h - k \leq \phi$  on  $A(h)$ . Thus,  
 $(h - k)^p |A(h)| \leq \int_{A(h)} |\phi|^p dx \leq \int_{A(k)} |\phi|^p dx$ . Combine the two last inequalities to obtain

$$(h - k) |A(h)|^{1/p} \leq C \|f\|_{L^\infty} |A(k)|^{1/q}$$

or  $|A(h)| \leq \left( \frac{C \|f\|_{L^\infty}}{h - k} \right)^p |A(k)|^{p/q}$ , for  $q < p$ . With iteration technique we finally obtain that  
 $u^\epsilon \leq l + C 2^{p/p-q} |G^\epsilon(\omega)|^{\frac{p-q}{qp}} \|f\|_{L^\infty(G^\epsilon(\omega))}$  so that

$$\sup_{G^\epsilon(\omega)} u^\epsilon \leq \sup_{\partial G^\epsilon(\omega)} u^\epsilon + C \|f^\epsilon\|_{L^\infty(\Omega)} \quad (5.2.5)$$

As corollary, we also have

$$\inf_{G^\epsilon(\omega)} u^\epsilon \geq \inf_{\partial G^\epsilon(\omega)} u^\epsilon - C \|f^\epsilon\|_{L^\infty(\Omega)} \quad (5.2.6)$$

Standard estimation gives also Holder continuity of the sequence  $u^\epsilon$ :

$$\|u^\epsilon\|_{C^\alpha(G^\epsilon(\omega))} \leq C \|f\|_{L^\infty(G^\epsilon(\omega))} \quad (5.2.7)$$

Using remark (5.2.5) and the developed extension  $\tilde{u}^\epsilon$  converges up to a subsequence, still denoted by  $\tilde{u}^\epsilon$ , to a function  $u \in W^{1,2}(D) \cap C^\alpha(D)$ .

Its enough now to show that the limit  $u$  is the minimizer of the indicated integral.

Step1: We first establish the inequality  $\limsup_{\epsilon \rightarrow 0} J^\epsilon[u^\epsilon] \leq J[w]$  for all  $w \in W^{1,2}(D)$ .

Consider a partition of  $D$  with cubes  $K^\alpha = K(x^\alpha, h)$  centered at  $x^\alpha$  of size  $h$ , so that  $\cup_\alpha K(x^\alpha, h)$  is a cover of  $D$  and the nodes  $x^\alpha$  form a periodic lattice of period  $h - r$ ,  $r$  to be chosen. Consider a partition of unity  $\{\phi_\alpha\}$  of  $C^2$  functions such that

1.  $0 \leq \phi_\alpha \leq 1$
2.  $\phi_\alpha = 0$  if  $x \notin K^\alpha$ ,  $\phi_\alpha = 1$  if  $x \in K^\alpha \setminus \cup_{\beta \neq \alpha} K^\beta$
3.  $\sum_{\alpha} \phi_\alpha(x) = 1$ , if  $x \in D$
4.  $|\nabla \phi_\alpha| \leq C/r$

For every unit vector  $\xi_i = e_i, i = 1, \dots, n$  let  $v_i^\alpha$  be the corresponding minimizer of  $P_{\epsilon, h}^z(\xi)$ . For  $w \in C^2(D)$ , compactly supported in  $\Omega$ , define

$$w_h^\epsilon(x) = w(x) + \sum_{\alpha} \sum_{i=1}^n \frac{\partial w}{\partial x_i} [v_i^\alpha - (x_i - x_i^\alpha)] \phi_\alpha(x) \quad (5.2.8)$$

so that  $w_h^\epsilon \in W^{1,2}(D)$ .

Since  $u^\epsilon$  is the minimizer,

$$J^\epsilon[u^\epsilon] \leq J^\epsilon[w_h^\epsilon] \quad (5.2.9)$$

The right hand side of (5.2.9) is estimated as follows: Having proved the existence of the limit of  $\{a_{ij}\}$ , we have

$$\int_{K^\alpha \cap G^\epsilon(\omega)} |\nabla v_i^\alpha|^2 + h^{-2-\gamma} |v_i^\alpha - (x_i - x_i^\alpha)|^2 dx \leq Ch^n$$

which implies that

$$\int_{K^\alpha \cap G^\epsilon(\omega)} |\nabla v_i^\alpha|^2 dx \leq Ch^n \quad (5.2.10)$$

$$\int_{K^\alpha \cap G^\epsilon(\omega)} |v_i^\alpha - (x_i - x_i^\alpha)|^2 dx \leq Ch^{n+2+\gamma}$$

Take now the concentric cube  $K_1^\alpha = K^\alpha \setminus \cup_{\beta \neq \alpha} K^\beta$  of size  $h - 2r$ . Then

$$\begin{aligned} & \int_{(K^\alpha \setminus K_1^\alpha) \cap G^\epsilon(\omega)} |\nabla v_i^\alpha|^2 + h^{-2-\gamma} |v_i^\alpha - (x_i - x_i^\alpha)|^2 dx \\ &= \int_{(K^\alpha \cap G^\epsilon(\omega))} |\nabla v_i^\alpha|^2 + h^{-2-\gamma} |v_i^\alpha - (x_i - x_i^\alpha)|^2 dx \\ & - \int_{K_1^\alpha \cap G^\epsilon(\omega)} |\nabla v_i^\alpha|^2 + h^{-2-\gamma} |v_i^\alpha - (x_i - x_i^\alpha)|^2 dx + O(rh^{n-1}) \\ & \leq a_{ii}(x^\alpha, \epsilon, h, \gamma, \omega) - a_{ii}(x^\alpha, \epsilon, h_1, \gamma, \omega) + O(rh^{n-1}) \end{aligned}$$

where  $h_1 = h - 2r$ . Thus, if  $r = h^{1+\gamma/2} = O(h)$  we obtain

$$\int_{(K^\alpha \setminus K_1^\alpha) \cap G^\epsilon(\omega)} |\nabla v_i^\alpha|^2 dx = O(h^n)$$

and

$$\int_{(K^\alpha \setminus K_1^\alpha) \cap G^\epsilon(\omega)} |v_i^\alpha - (x_i - x_i^\alpha)|^2 dx = O(h^{2+n+\gamma})$$

Next, we compute the derivatives of  $w_h^\epsilon$ :

$$\frac{\partial w_h^\epsilon}{\partial x_j} = \sum_\alpha \sum_{i,j} \left[ \frac{\partial^2 w}{\partial x_i \partial x_j} [v_i^\alpha - (x_i - x_i^\alpha) \phi_\alpha] + \frac{\partial w}{\partial x_i} \frac{\partial v_i^\alpha}{\partial x_j} \phi_\alpha + \frac{\partial w}{\partial x_i} \frac{\partial \phi_\alpha}{\partial x_j} [v_i^\alpha - (x_i - x_i^\alpha)] \right]$$

Substituting  $w_h^\epsilon$  and its derivatives in  $J^\epsilon[\cdot]$ , we have

$$\begin{aligned} J^\epsilon[w_h^\epsilon] = & \sum_\alpha \sum_{i,j} \int_{K^\alpha \cap G^\epsilon(\omega)} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \nabla v_i^\alpha \cdot \nabla v_j^\alpha dx + \int_{F^\epsilon(\omega)} \lambda w^2 + 2f^\epsilon w dx + \\ & \sum_{\alpha,\beta} \left( \sum_{i,j} \int_{(K^\alpha \cap K^\beta) \cap G^\epsilon(\omega)} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \nabla v_i^\alpha \cdot \nabla v_j^\beta [\phi_\alpha \phi_\beta - \delta_{\alpha,\beta}] dx + E_{\alpha,\beta}(\epsilon, h, r, \omega) \right) \end{aligned} \quad (5.2.11)$$

where  $E_{\alpha,\beta}$  are the quadratic and linear combinations of  $[v_i^\alpha - (x_i - x_i^\alpha) \phi_\alpha]$ ,

$[v_i^\beta - (x_i - x_i^\beta) \frac{\partial \phi_\beta}{\partial x_j}]$  over  $K^\alpha \cap K^\beta \cap G^\epsilon(\omega)$ . Note that

$$\limsup_{\epsilon \rightarrow 0} \sum_{\alpha,\beta} |E_{\alpha,\beta}(\epsilon, h, r, \omega)| = o(r^{-1} h^{1+\gamma/2}) \text{ as } h \rightarrow 0 \text{ so that, since } r = h^{1+\gamma/2},$$

$$\lim_{h \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \sum_{\alpha,\beta} |E_{\alpha,\beta}(\epsilon, h, r, \omega)| = 0.$$

Now we estimate

$$\begin{aligned} & \sum_{i,j} \int_{K^\alpha \cap G^\epsilon(\omega)} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \nabla v_i^\alpha \cdot \nabla v_j^\alpha dx \\ & \leq \sum_{i,j} \int_{K^\alpha \cap G^\epsilon(\omega)} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \nabla v_i^\alpha \cdot \nabla v_j^\alpha + h^{-2-\gamma} (v_i^\alpha - (x_i - x_i^\alpha))(v_j^\alpha - (x_j - x_j^\alpha)) dx \\ & \leq \sum_{i,j} \frac{\partial w}{\partial x_i}(x^\alpha) \frac{\partial w}{\partial x_j}(x^\alpha) a_{ij}(x^\alpha, \epsilon, \gamma, \omega) + O(h^{n+1}) \end{aligned}$$

for  $\epsilon$  small enough.

Finally we observe that

$$\lim_{h \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \sum_{\alpha, \beta} \sum_{i, j} \int_{K^\alpha \cap K^\beta \cap G^\epsilon(\omega)} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \nabla v_i^\alpha \cdot \nabla v_j^\alpha [\phi_\alpha \phi_\beta - \delta_{\alpha, \beta}] dx = 0$$

The conclusion of these estimates is that

$$\limsup_{\epsilon \rightarrow 0} J^\epsilon(u^\epsilon) \leq \lim_{h \rightarrow 0} \limsup_{\epsilon \rightarrow 0} J^\epsilon(w_h^\epsilon) = \int_D \sum_{i, j} a_{ij}(x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} + \lambda \beta(x) w^2 + 2f\beta(x) w dx \quad (5.2.12)$$

for any  $w \in C_0^2(D)$ . Thanks to the density of smooth functions in  $W^{1,2}(D)$ , the estimate holds for all  $w \in W_0^{1,2}(D)$ .

Step 2: We show the reverse inequality, i.e.

$$\liminf_{\epsilon \rightarrow 0} J^\epsilon[u^\epsilon] \geq J[u] \quad (5.2.13)$$

Pick  $u^\delta \in C^2(D)$  with  $\|u^\delta - u\|_{W^{1,2}(D)} < \epsilon$  and let  $\tilde{u}_\delta^\epsilon = \tilde{u}^\epsilon + u^\delta - u$ , where  $\tilde{u}^\epsilon$  is the extension of  $u^\epsilon$  in  $D$ . Also, let  $u_\delta^\epsilon = u^\epsilon + u^\delta - u = \tilde{u}_\delta^\epsilon|_{G^\epsilon(\omega)}$ . Under these definitions,  $\tilde{u}_\delta^\epsilon \rightarrow u^\delta$  in  $L^2(D)$  and  $\|u_\delta^\epsilon - u^\epsilon\|_{W^{1,2}(G^\epsilon(\omega))} \leq \epsilon$ .

Divide the space into disjoint cubes  $K_h^\alpha = K(x^\alpha, h)$  centered at the points  $x^\alpha$  and let  $v_\delta^{\alpha, \epsilon}(x) = u_\delta^\epsilon(x) - u_\delta(x^\alpha)$  for every cube with  $K_h^\alpha \cap G^\epsilon(\omega) \neq \emptyset$ . Since  $u^\delta \in C_0^2(D)$ ,  $\forall \delta > 0$  and  $\xi \in \mathbb{R}^n$ ,

$$\begin{aligned} & \int_{K_h^\alpha \cap G^\epsilon(\omega)} |v_\delta^{\alpha, \epsilon}(x) - (x - x^\alpha, \xi)|^2 dx \\ & \leq \int_{K_h^\alpha \cap G^\epsilon(\omega)} |u_\delta^\epsilon(x) - u_\delta|^2 dx + \int_{K_h^\alpha \cap G^\epsilon(\omega)} [\nabla u_\delta(x^\alpha) \cdot (x - x^\alpha) - (x - x^\alpha, \xi)]^2 dx + O(h^{n+4}) \end{aligned}$$

Choose  $\xi_\alpha = \nabla u_\delta(x^\alpha)$ : Then

$$\limsup_{\epsilon \rightarrow 0} \int_{K_h^\alpha \cap G^\epsilon(\omega)} |v_\delta^{\alpha, \epsilon}(x) - (x - x^\alpha, \nabla u_\delta(x^\alpha))|^2 dx = O(h^{n+4})$$

On the other hand,

$$\begin{aligned} \int_{K_h^\alpha \cap G^\epsilon(\omega)} \{ |\nabla v_\delta^{\alpha, \epsilon}(x)|^2 + h^{-2-\gamma} |v_\delta^{\alpha, \epsilon} - (x - x^\alpha, \xi_\alpha)|^2 \} dx \\ \geq \sum_{i,j=1}^n a_{ij}(x^\alpha, \epsilon, h, \gamma, \omega) \frac{\partial u_\delta}{\partial x_i}(x^\alpha) \frac{\partial u_\delta}{\partial x_j}(x^\alpha) \end{aligned}$$

which gives

$$\int_{K_h^\alpha \cap G^\epsilon(\omega)} |\nabla u_\delta^\epsilon|^2 \geq \sum_{i,j} a_{ij}(x^\alpha, h, \epsilon, \gamma, \omega) \frac{\partial u_\delta}{\partial x_i}(x^\alpha) \frac{\partial u_\delta}{\partial x_j}(x^\alpha) - O(h^{(n+2)-\gamma})$$

for  $\alpha = 1, \dots, N$ ,  $N = N(h) = O(h^{-n})$ .

Hence,

$$\begin{aligned} J^\epsilon[u^\epsilon - \delta] &\geq \sum_{\alpha=1}^N \int_{K_h^\alpha \cap G^\epsilon(\omega)} |\nabla u_\delta^\epsilon|^2 + \int_{G^\epsilon(\omega)} \lambda |u_\delta^\epsilon|^2 + 2f^\epsilon u_\delta^\epsilon dx \\ &\geq \sum_{\alpha=1}^N \sum_{i,j=1}^n a_{ij}(x^\alpha, h, \epsilon, \gamma, \omega) \frac{\partial u_\delta}{\partial x_i}(x^\alpha) \frac{\partial u_\delta}{\partial x_j}(x^\alpha) \\ &\quad + \int_{\Omega} \chi_{G^\epsilon(\omega)} |\tilde{u}_\delta^\epsilon|^2 + 2f^\epsilon \tilde{u}_\delta^\epsilon dx - O(h^{2-\gamma}) \end{aligned}$$

Fix  $\delta$ , and let  $\epsilon \rightarrow 0$ ,  $h \rightarrow 0$  to obtain  $\liminf_{\epsilon \rightarrow 0} J^\epsilon[u_\delta^\epsilon] \geq J[u_\delta]$ . Finally let  $\delta \rightarrow 0$ .

Thus  $u$  is the minimizer of  $J[\cdot]$ . This completes the proof.  $\square$

#### 5.2.4 Justification of the tensor limit

The justification of the mean conductivity tensor  $[a_{ij}]$  can be obtained from the following scheme:

For an arbitrary point  $x \in D$  and for  $h > 0$  sufficiently small so that the cube  $Q_h^x = Q(x, h)$  is in  $D$ , we prescribe constant temperature  $\pm \frac{h}{2}$  on two parallel faces  $\Gamma^+$ ,  $\Gamma^-$  of the cube. Let  $l$  be the vector perpendicular to  $\Gamma^+$ ,  $\Gamma^-$  passing through the center of the cube. Suppose that the other faces are insulated without flux.

Define the conductivity in the direction  $l$  by  $C_l(x, h) = \frac{1}{h^2} \int_{\Gamma^+ \cap G^\epsilon(\omega)} \frac{\partial u^\epsilon}{\partial l} dS$ .



In  $Q_h^x \cap G^\epsilon(\omega)$ , the temperature  $u^\epsilon$  is a harmonic function which solves the boundary value problem

$$\begin{aligned} -\Delta u^\epsilon &= 0, \quad x \in G^\epsilon(\omega) \\ \frac{\partial u^\epsilon}{\partial \eta} &= 0, \quad x \in \partial F^\epsilon(\omega) \cap Q_h^x, \quad x \in ((\partial Q_h^x \setminus \Gamma^+ \cup \Gamma^-) \cap G^\epsilon(\omega)) \\ u^\epsilon &= \pm \frac{h}{2}, \quad x \in \Gamma^\pm \cap G^\epsilon(\omega) \end{aligned} \quad (5.2.14)$$

and minimizes the functional  $J_{hl}^\epsilon(u^\epsilon) = \int_{Q_h^x \cap G^\epsilon(\omega)} |\nabla u^\epsilon|^2 dx$  over all  $v^\epsilon \in W^{1,2}(Q_h^x \cap G^\epsilon(\omega))$  equal to  $\pm \frac{h}{2}$  on  $\Gamma^\pm \cap G^\epsilon(\omega)$ . From the compactness argument carried out in section 5.2.2, up to a subsequence, there exists function  $u_l^h \in W^{1,2}(\Omega) \cap C^\alpha(\Omega)$  such that  $\|u^\epsilon - u_l^h\|_{L^2(\Omega)} \rightarrow 0$  and  $\|u^\epsilon - u_l^h\|_{C^\alpha(\Omega)} \rightarrow 0$  as  $\epsilon \rightarrow 0$ .

The computations of section 5.2.3 give us the inequalities

$$\limsup_{\epsilon \rightarrow 0} \int_{Q_h^x \cap G^\epsilon(\omega)} |\nabla u^\epsilon|^2 dx \leq \int_{Q_h^x \cap D} a_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} dx$$

and

$$\liminf_{\epsilon \rightarrow 0} \int_{Q_h^x \cap G^\epsilon(\omega)} |\nabla u^\epsilon|^2 dx \geq \int_{Q_h^x \cap D} a_{ij} \frac{\partial u_l^h}{\partial x_i} \frac{\partial u_l^h}{\partial x_j} dx$$

for all  $w \in W^{1,2}(Q_h^x \cap D)$ .

This implies that  $u_l^h$  is the corresponding minimizer and solves the BVP

$$\begin{aligned} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u_l^h}{\partial x_j} \right) &= 0, \quad x \in Q_h^x \cap D \\ \frac{\partial u_l^h}{\partial \eta} &= 0, \quad x \in \partial Q_h^x \setminus (\Gamma^+ \cup \Gamma^-) \\ u_l^h &= \pm \frac{h}{2}, \quad x \in \Gamma^\pm \end{aligned} \quad (5.2.15)$$

as long as  $a_{ij}$  is continuous. Thus,  $\|\nabla u_l^h - l\|_{L^2(Q_h^x)} = o(h^3)$ . Combining the inequalities, we

obtain  $\lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} C_l(x, h) = \sum_{i=1}^3 a_{ij}(x) l_i l_j$ .

This justification also corresponds to the rigorous proof of the existence of the limit on section 5.2.1.

### 5.3 Homogenization on sets of decreasing volume

Our goal now is to consider the Neumann problem (5.2.1) on  $F^\epsilon(\omega)$ . For such problems, we need a different definition for the convergence of solutions which is valid for sets for which, as  $\epsilon \rightarrow 0$   $|F^\epsilon(\omega)| \rightarrow 0$ .

To cover the general case for our broad class of random domains, we will use the notion of convergence due to Khruslov [12]. Before we go to that part, we mention that the convergence process is similar to the one used in the proof of theorem 5.2.8, as long as we define the following local characteristics:

1. The volume capacity

$$\beta(z, h, \epsilon, \omega) = \frac{\text{mes}(\Omega)}{\text{mes}(F^\epsilon(\omega))} \int_{F^\epsilon(\omega) \cap K_h^z} \chi_{F^\epsilon(\omega)} dx = \frac{\text{mes}(\Omega)}{\text{mes}(F^\epsilon(\omega))} \text{mes}(F^\epsilon(\omega) \cap K_h^z) \quad (5.3.1)$$

2. The functional

$$P_{\epsilon, h}^z(\xi) = \frac{\text{mes}(\Omega)}{\text{mes}(F^\epsilon(\omega))} \inf_{v^\epsilon \in W^{1,2}(K_h^z \cap F^\epsilon(\omega))} \int \{|\nabla v^\epsilon|^2 + h^{-2-\gamma} |v^\epsilon - (x - z, \xi)|^2\} dx \quad (5.3.2)$$

To proceed to the convergence, we note that regularity theory for elliptic equations implies that  $u^\epsilon \in C^\alpha(F^\epsilon(\omega)) \cap W^{1,2}(F^\epsilon(\omega))$ . Whitney's theorem then implies that  $u^\epsilon$  can be extended from  $F^\epsilon(\omega)$  to a function  $u_L^\epsilon$  in  $\Omega$  in a way that the extended sequence of functions is still Lipschitz and  $|u_L^\epsilon| \leq CL$ , where  $C$  is independent of  $L$  and  $\epsilon$ . Arzela's theorem implies now the existence of a subsequence  $u_L^\epsilon$  that converges uniformly in  $\Omega$  to some  $u_L$  in  $L^2(\Omega)$ . Assuming that, for sufficient density of points, the following condition holds: for any ball  $B_r$  of radius  $r > 0$  and for  $\epsilon > 0$  large enough,

$$C_1 \epsilon^3 \text{mes}(F^\epsilon(\omega)) \leq \text{mes}(F^\epsilon(\omega) \cap B_r) \leq C_2 \epsilon^3 \text{mes}(F^\epsilon(\omega)).$$

Then, one can show that the sequence  $\{u_L\}, L = 1, 2, \dots$  is Cauchy and thus  $u_L$  converges to some  $u \in L^2(\Omega)$ . Then,  $\lim_{L \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \frac{1}{|F^\epsilon(\omega)|} \|u^\epsilon - u_L\|_{L^2(F^\epsilon(\omega))}^2 = 0$ . This notion of convergence allow us to deal with problems of decreasing volume.

**Theorem 5.3.1.** *The (extended) sequence of solutions  $u^\epsilon$  of problems (5.2.1), defined in  $F^\epsilon(\omega)$  converge to the solution  $u$  of the boundary value problem*

$$\begin{aligned} -\frac{1}{\beta} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \lambda u &= f, \quad x \in D \\ \frac{\partial u^\epsilon}{\partial \eta} &= 0 \quad x \in \partial D \end{aligned} \tag{5.3.3}$$

in the sense that  $\lim_{L \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \frac{1}{|F^\epsilon(\omega)|} \|u^\epsilon - u\|_{L^2(F^\epsilon(\omega))}^2 = 0$

The proof follows closely the one of theorem 5.2.8.

## 5.4 Time dependent problems

Let  $F^\epsilon(\omega) = \cup_{x_i \in X} B(x_i, \rho^\epsilon(\omega))$  be the set of non-intersecting random spheres of the Boolean model and let  $G^\epsilon(\omega) = \mathbb{R}^n \setminus F^\epsilon(\omega)$ . From the definition,  $\text{mes} F^\epsilon(\omega) \rightarrow 0$  as  $\epsilon \rightarrow 0$  and  $G^\epsilon(\omega)$  is connected.

Consider the diffusion equation

$$\frac{\partial u^\epsilon(x, t)}{\partial t} - \Delta u^\epsilon(x, t) = 0, \quad x \in G^\epsilon(\omega) \tag{5.4.1}$$

$$\frac{\partial u^\epsilon}{\partial \eta} = 0, \quad x \in \partial F^\epsilon(\omega), \quad t > 0 \tag{5.4.2}$$

$$u^\epsilon = f^\epsilon(x, t), \quad x \in \partial D, \quad t > 0 \tag{5.4.3}$$

$$u^\epsilon(x, 0) = \phi(x) \tag{5.4.4}$$

We look for the asymptotic behavior of (5.4.1) – (5.4.4) as  $\epsilon \rightarrow 0$ . To obtain the average equations, one has to consider the stationary problem for fixed time  $t > 0$ . It turns out that the sequence  $u^\epsilon(x, t)$  converges to the solution  $u(x, t)$  of the following average problem:

$$\begin{aligned}
\frac{\partial u}{\partial t} - \frac{1}{\beta}(\bar{a}\Delta u) &= 0, \quad x \in D, \quad t > 0 \\
u &= f(x, t), \quad x \in \partial D, \quad t > 0 \\
u^\epsilon(x, 0) &= \phi(x), \quad x \in D
\end{aligned} \tag{5.4.5}$$

where  $\bar{a}$ ,  $\beta$  are the limits of local characteristics of the microstructure, as long as  $f^\epsilon \rightarrow f$  in the sense that  $\lim_{\epsilon \rightarrow 0} \|f^\epsilon - f\|_{L^\infty(D)} = 0$

#### 5.4.1 Smoothness of $u^\epsilon$ and compactness

As in the case of elliptic equations, we discuss first the smoothness and compactness of solutions of the problems (5.4.1) – (5.4.4) in strongly perforated domains  $G^\epsilon(\omega)$ .

Fix  $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^+$ , with  $t_0 > R^2$ . Let

$$B_R = B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\},$$

$$Q_R = Q_R(x_0, t_0) = B_R \times (t_0 - R^2, t_0),$$

$$Q_{\alpha R} = Q_{\alpha R}(x_0, t_0) = B_{\alpha R} \times (t_0 - (1 + \alpha)R^2, t_0).$$

The following lemma is a version of Poincaré's inequality:

**Lemma 5.4.1.** *Suppose  $D \subset \mathbb{R}^n$  is bounded, convex,  $G \subset D$ , and  $u \in W^{1,p}(D)$ , with  $1 \leq p \leq \infty$ .*

*If  $u_G = \frac{1}{|G|} \int_G u(x) dx$ , then  $\|u - u_G\|_{L^p(D)} \leq \frac{C}{|G|} (\text{diam } D)^{n+1} \|\nabla u\|_{L^p(D)}$ .*

**Lemma 5.4.2.** *For any  $\gamma > 0$ , there exists function  $g(z) \in C^2(0, \infty)$  such that*

$$i) \quad \forall z > 0, \quad g''(z) \geq [g'(z)]^2 - \gamma g'(z), \quad g'(z) \leq 0$$

$$ii) \quad \text{as } z \rightarrow 0^+, \quad g(z) \approx -\ln z$$

$$iii) \quad g(z) = 0 \quad \text{for } z \geq 1$$

$$iv) \quad \text{the function } G(z) = g(az + b) \text{ satisfies (i) for } a \geq 1, \quad b > 0.$$

*Proof.* First, note that the function  $\hat{g}(z) = \left(-\ln \frac{1 - e^{-\gamma z}}{1 - e^{-\gamma}}\right)^+$ ,  $z > 0$  is nonincreasing since

$$\hat{g}'(z) = \frac{-\gamma e^{-\gamma z}}{1 - e^{-\gamma z}}. \quad \text{Also it satisfies (ii)-(iii).}$$

Now if  $\hat{f}(z) = -e^{-\hat{g}(z)} = \max\{-1, -\frac{1 - e^{-\gamma z}}{1 - e^{-\gamma}}\}$  then  $\hat{f}''(z) + \gamma \hat{f}'(z) = 0$  for  $z \neq 1$  and  $\hat{f}(z)$  satisfies (i)-(iii) except at  $z = 1$ .

One can use a smooth approximation of  $\hat{g}(z)$ : If  $\hat{g}_0(z) = \hat{f}'(z) + \gamma\hat{f}(z)$ , then

$$\hat{g}_0(z) = \begin{cases} -\frac{\gamma}{1-e^{-\gamma}}, & \text{for } 0 \leq z < 1 \\ -\gamma, & \text{for } 1 \leq z < \infty \end{cases}$$

and  $\lim_{z \rightarrow 1^+} \hat{g}_0(z) > \lim_{z \rightarrow 1^-} \hat{g}_0(z)$ .

A straightforward computation gives  $\int_0^2 e^{\gamma z} \hat{g}_0(z) dz = -e^{2\gamma}$ .

Now we can find a smooth approximation  $g_0(z)$  of  $\hat{g}_0(z)$  such that  $g_0 \in C^\infty[0, \infty)$ ,  $g_0(z) < 0$ ,  $g'_0(z) > 0$  and  $g_0(z) = \hat{g}_0(z)$  for  $z \in [0, 1/2] \cup [2, \infty)$ , and  $\int_0^2 e^{\gamma z} g_0(z) dz = -e^{2\gamma}$ .

Then we can solve the equation  $f'(z) + \gamma f(z) = g_0(z)$  with  $f(0) = 0$ . Hence,

$f(z) = e^{-\gamma z} \int_0^z e^{\gamma z} g_0(z) dz$ ,  $z > 0$ , and  $f(z) = -1$  if  $z \geq 2$ . Finally, let  $g(z) = -\ln(-f(z))$ ,  $z > 0$ . (iv) follows. □

**Proposition 5.4.3.** *Suppose that  $u^\epsilon \in H_{loc}^1(\mathbb{R}^n \times \mathbb{R}^+)$  is a nonnegative solution of (5.4.1).*

*Assume that  $|\{(x, t) \in Q_R : u(x, t) \geq 1\}| \geq \theta |Q_R|$  with  $\theta \in (0, 1)$ . Then for all  $\sigma \in (0, \theta)$ ,  $\beta \in (\theta, 1)$  such that  $\frac{1-\theta}{1-\sigma} = \frac{2\beta^n}{3}$ , there exists  $h = h(n, \theta)$  so that*

$$|\{x \in B_{\beta R} : u(x, t) \geq h\}| \geq \frac{1}{4} |B_{\beta R}| \text{ for all } t \in [t_0 - \sigma R^2, t_0]. \quad (5.4.6)$$

*Proof.* We choose a cutoff function  $\zeta \in C_0^\infty(B_R)$  such that  $0 \leq \zeta \leq 1$ ,  $\zeta = 1$  on  $B_{\beta R}$  and  $\nabla \zeta \leq \frac{c}{(1-\beta)R}$ . In the weak formulation  $\int_{t_1}^{t_2} \int_{B_R} u_t \phi + \nabla u \nabla \phi dx dt = 0$ , we choose  $\phi = G(u) \zeta^2 \chi_{[t_1, t_2]}$  as test function, where  $\chi_{[t_1, t_2]}$  is the characteristic function of the set  $[t_1, t_2]$ , for arbitrary  $t_0 - R^2 \leq t_1 \leq t_2 \leq t_0$ . After differentiation, we have

$$\int_{t_1}^{t_2} \int_{B_R} \zeta^2 G'(u) \chi_{[t_1, t_2]} u_t + \zeta^2 G''(u) |\nabla u|^2 + G'(u) 2\zeta \nabla \zeta \cdot \nabla u dx dt = 0.$$

Thanks to the Holder inequality  $2\zeta |\nabla w \cdot \nabla \zeta| \leq \frac{1}{2} \zeta^2 |\nabla w|^2 + 2 |\nabla \zeta|^2$ , where  $w = G(u)$ , we obtain

$$\int_{t_1}^{t_2} \int_{B_R} \zeta^2 w_t dx dt + \frac{1}{2} \int_{t_1}^{t_2} \int_{B_R} \zeta^2 |\nabla w|^2 dx dt \leq 2 \int_{t_1}^{t_2} \int_{B_R} |\nabla \zeta|^2 dx dt \leq C |B_R| \quad (5.4.7)$$

Let now  $A(t) = |\{x \in B_R : u(x, t) \geq 1\}|$ ,  $B(t) = |\{x \in B_{\beta R} : u(x, t) \geq h\}|$ , for  $h$  to be determined. By assumption,  $\int_{t_0-R^2}^{t_0} A(t) dt \geq \theta R^2 |B_R|$  and clearly,

$$\int_{t_0-\sigma R^2}^{t_0} A(t) dt \leq \sigma R^2 |B_R|.$$

Then,  $\int_{t_0-R^2}^{t_0-\sigma R^2} A(t)dt \geq (\mu - \sigma)R^2|B_R|$ . The mean value theorem implies that there is  $\tau \in [t_0 - R^2, t_0 - \sigma R^2]$  such that  $A(\tau) \geq \frac{\theta - \sigma}{1 - \sigma}|B_R|$ . Then for  $\tau = t_1$ ,

$$\int_{\tau}^{t_2} \int_{B_R} \zeta^2 w_t dx dt \leq C(\theta)|B_{\beta R}| \text{ so that after a time-integration,}$$

$$\int_{B_R} \zeta^2 w(x, t_2) dx \leq C(\theta)|B_{\beta R}| + \int_{B_R} \zeta^2 w(x, \tau) dx \quad (5.4.8)$$

with  $w = g(u + h)$ ,  $g'(z) \leq 0$ .

$$\text{Then, } \int_{B_R} \zeta^2 w(x, t_2) dx \geq \int_{B_{\beta R} \setminus B(t_2)} w(x, t_2) dx \geq |B_{\beta R} \setminus B(t_2)|g(2h)$$

Also,

$$\begin{aligned} \int_{B_R} \zeta^2 w(x, t) dx &\leq \int_{B_R} w(x, t) dx = \int_{\{x \in B_R : 1 > u^\epsilon(x, t)\}} w(x, t) dx \leq (|B_R| - A(t))g(h) \leq \\ (1 - \frac{\theta - \sigma}{1 - \sigma})g(h)|B_R| &= \frac{(1 - \theta)}{(1 - \sigma)\beta^n}g(h)|B_{\beta R}| \end{aligned} \quad (5.4.9)$$

Combining the last inequalities, we obtain that

$$|B_{\beta R} \setminus B(t)| \leq \frac{2g(h) + C}{3g(2h)}|B_{\beta R}|$$

Finally, since  $g(z) \approx -\ln z$  as  $z \rightarrow 0^+$ , we can choose  $h$  small enough, these inequalities combined give us

$$|B_{\beta R} \setminus B(t_2)| \leq \frac{3}{4}|B_{\beta R}|, \text{ from which (5.4.6) follows.} \quad \square$$

**Lemma 5.4.4.** *Suppose that  $u^\epsilon \in H^1(\mathbb{R}^n \times \mathbb{R}^+)$  is a weak solution of (5.4.1) such that*

*$|\{x \in B_{\beta R} : u^\epsilon(x, t) \geq h\}| \geq \mu|B_{\beta R}|$  for all  $t \in [t_0 - \sigma R^2, t_0]$ , some  $\mu \in (0, 1)$ . Then there is*

*$\gamma > 0$ ,  $\gamma = \gamma(n, \mu, h)$  such that for  $\delta = 1/2 \min(\beta, \sqrt{\sigma})$ ,*

*$u^\epsilon(x, t) > \gamma$  for all  $(x, t) \in Q_{(\delta R)}$ .*

*Proof.* We denote by  $w = G(u^\epsilon)$ , with  $G'(s) \leq 0$ ,  $G'' - (G')^2 \geq 0$ . Then  $w$  is a subsolution of (5.4.1), since  $w_t - \Delta w = -G''(u^\epsilon)\nabla u^{\epsilon^2} \leq 0$ . The local boundedness of  $w$  gives

$$\sup_{Q(\delta R)} w^2 \leq \frac{C}{R^{n+2}} \|w\|_{L^2(Q_{2\delta R})}^2 \quad (5.4.10)$$

For  $0 < k < h$ , if  $w = G(u^\epsilon) = g((u^\epsilon + k)/h)$  with  $g$  as in lemma (5.4.2), then  $g(k) \geq w$  so that inequality (5.4.7) gives

$$\int_{t_0-\sigma R^2}^{t_0} \int_{B_R} (\zeta^2 w)_t dx dt + \frac{1}{2} \int_{t_0-\sigma R^2}^{t_0} \int_{B_R} \zeta^2 \nabla w^2 dx dt \leq cR^n \text{ and}$$

$$\int_{t_0-\sigma R^2}^{t_0} \int_{B_R} (\zeta^2 w)_t dx dt \geq -CR^n g(k/h).$$

Combining, we have

$$\int_{t_0-\sigma R^2}^{t_0} \int_{B_R} \nabla w^2 dx dt \leq CR^n (1 + g(k/h)) \quad (5.4.11)$$

Our assumption and the fact that  $g = 0$  for  $s \geq 1$  give us  $|\{x \in B_{\beta R} : w(x, t) = 0\}| \geq \mu |B_{\beta R}|$  for all  $t \in [t_0 - \sigma R^2, t_0]$ ,  $\mu \in (0, 1)$ .

Lemma (5.4.1) reads

$$\int_{B_{\beta R}} w^2(x, t) dx \leq \frac{C(\beta R)^{2n+2}}{|G|} \int_{B_{\beta R}} \nabla w(x, t)^2 dx \leq CR^2 \int_{B_{\beta R}} \nabla w(x, t)^2 dx, \text{ where}$$

$$G = \{x \in B_{\beta R} : w(x, t) = 0\}.$$

$$\text{Then, } \int_{t_0-\sigma R^2}^{t_0} \int_{B_R} w^2(x, t) dx dt \leq CR^{n+2} (1 + g(k/h)).$$

The last inequality with (5.4.9) says  $\sup_{Q(\delta R)} w^2 \leq \frac{C}{R^{n+2}} \|w\|_{L^2(Q_{2\delta R})}^2 \leq C(1 + g(k/h))$ . If  $\gamma < h/2$ , we must have  $u^\epsilon(x, t) > \gamma$  for all  $(x, t) \in Q_{(\delta R)}$ , otherwise we would have  $g^2(2\gamma/h) > C(1 + g(\gamma/h))$ , which is a contradiction.  $\square$

As a corollary we have the following theorem:

**Theorem 5.4.5.** *Suppose that  $u^\epsilon \in H^1(\mathbb{R}^n \times \mathbb{R}^+)$  is a weak solution of (5.4.1). If for some constants  $\mu \in (0, 1)$  and  $h > 0$   $|\{(x, t) \in Q_R : u^\epsilon(x, t) \geq h\}| \geq \mu |Q_R|$  holds, then there is  $\delta \in (1/2, 1)$ ,  $\delta = \delta(n, \mu)$  and  $\gamma = \gamma(n, \mu, h, \delta) > 0$  such that*

$$u^\epsilon(x, t) \geq \gamma \quad (5.4.12)$$

for all  $(x, t) \in Q_{\delta R}$ .

To establish Holder estimate, we define  $\text{osc}_{Q_R} u^\epsilon = \sup_{Q_R} u^\epsilon - \inf_{Q_R} u^\epsilon$ .

**Proposition 5.4.6.** *Suppose that  $u^\epsilon \in H^1(Q_{\tilde{R}})$  is a weak solution of (5.4.1) in  $Q_{\tilde{R}}$ . Then there are constants  $\delta \in (0, 1/2)$ ,  $\sigma \in (0, 1)$  that depend only on  $n$  such that for any  $0 < R < \tilde{R}/2$ , either*

$$i) \text{osc}_{Q_R} u^\epsilon \leq CR$$

or

$$ii) \operatorname{osc}_{Q_{\delta R}} u^\epsilon \leq \sigma \operatorname{osc}_{Q_R} u^\epsilon$$

*Proof.* Let  $M = M(R) = \sup_{Q_R} u^\epsilon$  and assume without loss of generality that  $\operatorname{osc}_{Q_R} u^\epsilon = 2M$ . If  $R > M$  then  $\operatorname{osc}_{Q_{\delta R}} u^\epsilon \leq \operatorname{osc}_{Q_R} u^\epsilon = 2M < CR$ .

For  $R \leq M$ , note that either  $|\{(x, t) \in Q_R : u^\epsilon(x, t) \geq 0\}| \geq \frac{1}{2}|Q_R|$ ,  
or  $|\{(x, t) \in Q_R : -u^\epsilon(x, t) \geq 0\}| \geq \frac{1}{2}|Q_R|$ .

Assuming the first inequality holds true, we have that

$$|\{(x, t) \in Q_R : 1 + \frac{u^\epsilon}{M} \geq 1\}| \geq \frac{1}{2}|Q_R|.$$

Theorem 5.4.5 implies that there is  $\gamma > 0$  such that  $1 + \frac{u^\epsilon}{M} \geq \gamma$  in  $Q_{\delta R}$ , or

$(\gamma - 1)M \leq u^\epsilon \leq M$  in  $Q_{\delta R}$ . Thus,

$$\operatorname{osc}_{Q_{\delta R}} u^\epsilon = \sup_{Q_{\delta R}} u^\epsilon - \inf_{Q_{\delta R}} u^\epsilon \leq M(1 - \gamma/2) = \sigma \operatorname{osc}_{Q_R} u^\epsilon. \text{ with } \sigma = 1 - \gamma/2.$$

□

The following iteration lemma, together with proposition 5.4.6 gives Holder estimate for the family  $u^\epsilon$ :

**Lemma 5.4.7.** *Suppose that  $w$  and  $f$  are increasing functions on an interval  $(0, R_0]$  and suppose that there are positive constants  $\alpha, \beta$  and  $\tau$  with  $\tau < 1$  and  $\beta < \alpha$  such that*

$$r^{-\beta} f(r) \leq s^{-\beta} f(s), \text{ if } 0 < s \leq r \leq R_0 \quad (5.4.13)$$

and

$$w(\tau r) \leq \tau^\alpha w(r) + f(r), \text{ if } 0 < r \leq R_0 \quad (5.4.14)$$

Then there is constant  $C = C(\alpha, \beta, \tau)$  such that

$$w(r) \leq C \left[ \left( \frac{r}{R_0} \right)^\alpha w(R_0) + f(r) \right], \text{ if } 0 < s \leq r \leq R_0 \quad (5.4.15)$$

*Proof.* See [11], pg.54.

□

Using the iteration lemma to the oscillation of  $u^\epsilon$ , we further obtain

**Theorem 5.4.8.** *Suppose that  $u^\epsilon \in H^1(\mathbb{R}^n \times \mathbb{R}^+)$  is a weak solution of (5.4.1). Then there is constant  $\alpha \in (0, 1)$  such that  $[u^\epsilon]_{\alpha, Q_R} \leq C$ ,  $C = C(n, Q_R)$ .*



The smoothness of  $u^\epsilon$  can be derived for any parabolic cylinder  $Q_{\epsilon,T}(\omega) = G^\epsilon(\omega) \times [0, T]$ . Boundedness of solutions in  $H^1$  follows from energy estimate. Consequently, the sequence  $u^\epsilon$  is compact in  $C^\alpha(\Omega \times [0, T])$  and weakly compact in  $W^{1,2}(\Omega \times [0, T])$ , after taking into account remark (5.2.4). The convergence of solutions follows with the same process, after considering the corresponding elliptic problem.

### 5.4.2 Convergence

Consider the problem

$$\begin{aligned} -\Delta u^\epsilon + u^\epsilon &= f^\epsilon, \quad x \in G^\epsilon(\omega) \\ \frac{\partial u^\epsilon}{\partial \eta} &= 0, \quad x \in \partial G^\epsilon(\omega) \cup D \end{aligned} \tag{5.4.16}$$

Theorem (5.2.8) implies then that the sequence  $u^\epsilon$  converges to the solution of the average problem

$$\begin{aligned} -\frac{1}{\beta}(a\Delta u + u) &= f, \quad x \in D \\ \frac{\partial u}{\partial \eta} &= 0, \quad x \in \partial D \end{aligned} \tag{5.4.17}$$

For  $\delta > 0$ ,  $\psi(t) \in C_0^\infty[\delta, T]$ ,  $\phi(x) \in C^\infty(D)$ , after integration by parts, we have

$$\int_D u_t^\epsilon \psi(t) \phi(x) dx = \int_D u^\epsilon \psi_t(t) \phi(x) dx.$$

For fixed  $t \in [\delta, T]$ , (5.4.13) provides the limit of the solutions  $u^\epsilon(x, t)$  to  $u(x, t)$ . Since  $t, \delta$  are arbitrary, (5.4.5) follows.

## APPENDIX A. Randomly heterogeneous materials

A random heterogeneous material is a realization of a random (stochastic) process. The mathematical configuration starts with a probability space  $\Omega$  and for each  $\omega \in \Omega$  we consider a random realization of a two-phase medium (for simplicity) over a bounded region  $G \subset \mathbb{R}^n$ . For time-independent problems, we define a structure function  $\xi(x; \omega)$  which characterizes statistically the medium. Clearly, for fixed  $\omega$ , this function either varies continuously or takes discrete values depending on the structure considered.

The realization  $\omega$  occupies randomly  $G$  and it splits it into two disjoint sets  $G_1(\omega)$ ,  $G_2(\omega)$ , so that  $G = G_1(\omega) \cup G_2(\omega)$ .

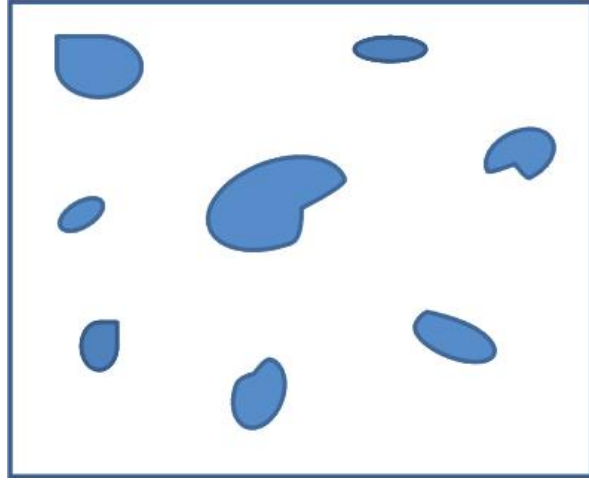


Figure A.1 A random two-phase heterogenous structure

If  $X^i(x, \omega)$ ,  $i = 1, 2$  is the indicator function at point  $x \in G$  of the phase  $G_i(\omega)$  respectively, then  $X^1(x, \omega) + X^2(x, \omega) = 1$  and  $P\{X^i(x, \omega) = 0\} = 1 - P\{X^i(x, \omega) = 1\}$ .

Let  $x_1, x_2, \dots, x_n$  be arbitrary points in  $G$  and consider the joint probability

$$P\{X^i(x_1) = 1, X^i(x_2) = 1, \dots, X^i(x_n) = 1\} \quad (\text{A.0.1})$$

that is, the probability to find all these points in phase  $i$ . This function is defined as the  $n$ -point probability function and has been introduced to determine effective properties of random media.

Under this context, we can define symmetric and ergodic properties for our structures.

**Definition A.0.9.** *The medium is called statistically homogeneous (or spatially stationary) if the joint probability distributions are translation invariant: there is vector  $y \in \mathbb{R}^n$  such that for all  $n \geq 1$  and groups of points  $x_1, x_2, \dots, x_n$  in  $\mathbb{R}^n$ ,*

$$P\{X^i(x_1) = 1, X^i(x_2) = 1, \dots, X^i(x_n) = 1\} = P\{X^i(x_1 + y) = 1, X^i(x_2 + y) = 1, \dots, X^i(x_n + y) = 1\}$$

Among other properties of statistically homogeneous media, we are interested in their ergodic properties: the fact that the joint probabilities depend on displacements and not on absolute positions suggests the ergodic hypothesis that one realization over infinite volume can give us the same information:

$$P\{X^i(x_1) = 1, X^i(x_2) = 1, \dots, X^i(x_n) = 1\} = \lim_{V \rightarrow \infty} \frac{1}{V} \int_V X^i(y) X^i(y + x_{12}) \dots X^i(y + x_{1n}) dy \quad (\text{A.0.2})$$

where  $x_{ij} = x_j - x_i$ .

For a more detailed presentation of heterogeneous materials, please see [17]. Continuum percolation models do have all these properties. Thus this theory combined with homogenization theory give certain homogenized equations for spatially stationary media.

## BIBLIOGRAPHY

- [1] M. A. Akcoglu and U. Krengel. Ergodic theorems for superadditive processes. *J. Reine Angew. Math.*, 323:53–67, 1981.
- [2] Grégoire Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. II. Noncritical sizes of the holes for a volume distribution and a surface distribution of holes. *Arch. Rational Mech. Anal.*, 113(3):261–298, 1990.
- [3] Grégoire Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.
- [4] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou. *Asymptotic analysis for periodic structures*, volume 5 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1978.
- [5] Alain Bourgeat, Andro Mikelić, and Steve Wright. Stochastic two-scale convergence in the mean and applications. *J. Reine Angew. Math.*, 456:19–51, 1994.
- [6] L. A. Caffarelli and A. Mellet. Random homogenization of an obstacle problem. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(2):375–395, 2009.
- [7] D. Cioranescu and F. Murat. Un terme étrange venu d’ailleurs. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980)*, volume 60 of *Res. Notes in Math.*, pages 98–138, 389–390. Pitman, Boston, Mass., 1982.
- [8] Doina Cioranescu and Jeannine Saint Jean Paulin. *Homogenization of reticulated structures*, volume 136 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999.

- [9] Gianni Dal Maso and Luciano Modica. Nonlinear stochastic homogenization. *Ann. Mat. Pura Appl. (4)*, 144:347–389, 1986.
- [10] Ulrich Hornung, editor. *Homogenization and porous media*, volume 6 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 1997.
- [11] Gary M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co. Inc., River Edge, NJ, 1996.
- [12] Vladimir A. Marchenko and Evgueni Ya. Khruslov. *Homogenization of partial differential equations*, volume 46 of *Progress in Mathematical Physics*. Birkhäuser Boston Inc., Boston, MA, 2006. Translated from the 2005 Russian original by M. Goncharenko and D. Shepelsky.
- [13] Ronald Meester and Rahul Roy. *Continuum percolation*, volume 119 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.
- [14] G. C. Papanicolaou and S. R. S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 835–873. North-Holland, Amsterdam, 1981.
- [15] Enrique Sánchez-Palencia. *Nonhomogeneous media and vibration theory*, volume 127 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, 1980.
- [16] Bo Su and Emmanuele DiBenedetto. Homogenization in the diffusion of ip3, preprint.
- [17] Salvatore Torquato. *Random heterogeneous materials*, volume 16 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 2002. Microstructure and macroscopic properties.